Truss Elements with Geometric Nonlinearity

Truss elements are important when considering geometrical nonlinearity, because they encompass the “leaning column” effect. Also, as shown in the excellent book by Crisfield, which appears in the reference list for this document, considering the truss element is a pedagogical introduction to geometric nonlinearity. Several strain measures are explained in the Crisfield book, but the Green-Lagrange strain is ultimately adopted, as in other documents on this website whenever geometric nonlinearity is included. That implies moderate but not large displacements, as well as small but not infinitesimally small strains. In addition to the choice of a strain measure, it is necessary to select a reference frame for the development of internal force and tangent stiffness for the element. Also explained by Crisfield, the options are Total Lagrangian, Updated Lagrangian, and the Corotational approach. All are addressed in the Crisfield book, while only the Updated Lagrangian approach is covered in this document and the accompanying implementation in Element 3 in the Python code G2 posted on this website. With the Updated Lagrangian approach, the starting point for the derivations is the truss element in its Local configuration shown in Figure 1. As indicated in the figure, that coordinate system is the orientation of the element in its last converged equilibrium position. That is the Updated Lagrangian approach.

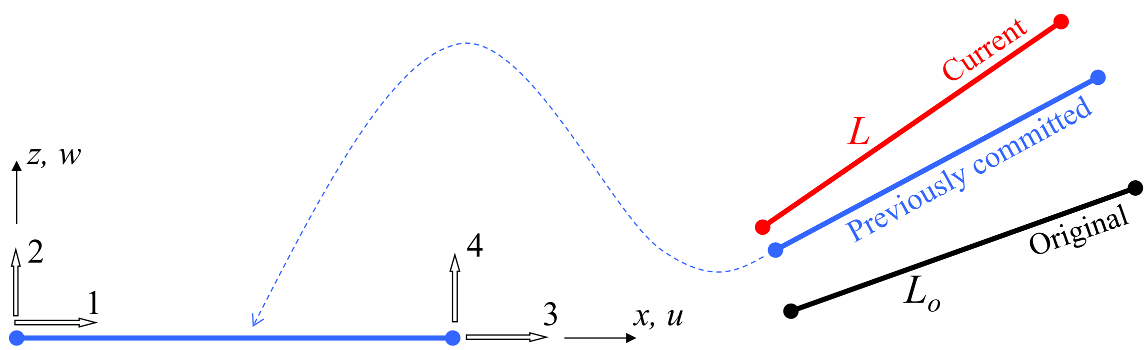

Figure 1: Axes, displacements, and degrees of freedom for 2D truss element.

Kinematic Compatibility

As shown in a separate document on Green-Lagrange strain, posted on this website, there are several equivalent ways to express that strain measure. The version adopted in the implementation of Element 3 in G2 is

$$\varepsilon = \frac{1}{2}\left(\left(\frac{L}{L_0}\right)^2 - 1\right) \tag{1}$$

where L_0 is the original element length and L is the current element length. Because L is calculated from the current displaced nodal coordinates, Eq. (1) captures the elongation and shortening that comes from rotation of the element. That is the essence of the geometric nonlinearity considered here. For the sake of subsequent developments, it is here emphasized that Eq. (1) is equivalent to

$$\varepsilon = \frac{du}{dx} + \frac{1}{2}\left(\frac{dw}{dx}\right)^2 \tag{2}$$

Material Law

In this document, the focus is on geometric nonlinearity. For that reason, the linear elastic material law is adopted. This implies small strains but still the possibility of moderate displacements:

$$\sigma = E \cdot \varepsilon \tag{3}$$

Shape Functions

The finite element method is, with few exceptions, displacement-based. That means that displacement interpolation, i.e., shape functions for the displaced shape of the element, is central. To include geometric nonlinearity for a 2D truss element it is necessary to consider all four degrees of freedom shown in Figure 1. This is an extended version of the "local" degrees of freedom of ordinary linear structural analysis. In the notes on this website, the displacement interpolation is generally written as

$$\tilde{\mathbf{u}} = \mathbf{N}\mathbf{u} \tag{4}$$

where $\tilde{\mathbf{u}}$=is the vector of displacement fields, $\mathbf{N}$=matrix with shape functions, and $\mathbf{u}$=degrees of freedom.

In the present case, Eq. (4) is spelled out as

$$\begin{Bmatrix} u(x) \\ w(x) \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 \\ 0 & N_1 & 0 & N_2 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} \tag{5}$$

where

$$N_1(x) = 1 - \frac{x}{L} \quad \text{and} \quad N_2(x) = \frac{x}{L} \tag{6}$$

Principle of Virtual Displacements

This principle expresses equilibrium in an average sense:

$$\int_V \sigma\, \delta\varepsilon\, dV = \int_0^L \mathbf{p}^T\, \delta\tilde{\mathbf{u}}\, dx \tag{7}$$

where $\mathbf{p}$ is a two-dimensional vector containing distributed loads in the x- and z-directions and $\tilde{\mathbf{u}}$ is defined in Eq. (5). Substitution of material law and integrating over the cross-section area yields

$$EA \cdot \int_0^L \varepsilon\, \delta\varepsilon\, dx = \int_0^L \mathbf{p}^T\, \delta\tilde{\mathbf{u}}\, dx \tag{8}$$

The virtual strain, $\delta\varepsilon$, requires special attention in face of geometric nonlinearity. In the linear case, matters are simple: $\delta\varepsilon=\delta u'$. That changes in light of the strain in Eq. (2). To address this issue, it is helpful to think of δ as a "variation" instead of "virtual." Calculus of variation, i.e., executing the variation, yields

$$\delta\varepsilon = \delta\left(u' + \frac{1}{2} \cdot w'^2 \right) = \delta u' + w' \cdot \delta w' \tag{9}$$

With $\delta\varepsilon$ addressed, at least for now, attention turns to ε, still in light of Green's strain in Eq. (2). The first objective is to express the derivatives du/dx and dw/dx in that strain expression in terms of shape functions. To that end, Eq. (5) is first split up, for convenience:

$$u(x) = \begin{bmatrix} N_1 & 0 & N_2 & 0 \end{bmatrix} \mathbf{u} \tag{10}$$

$$w(x) = \begin{bmatrix} 0 & N_1 & 0 & N_2 \end{bmatrix} \mathbf{u} \tag{11}$$

That means the sought derivatives are

$$\frac{du}{dx} = \begin{bmatrix} -\frac{1}{L} & 0 & \frac{1}{L} & 0 \end{bmatrix} \mathbf{u} \equiv \mathbf{Bu} \tag{12}$$

$$\frac{dw}{dx} = \begin{bmatrix} 0 & -\frac{1}{L} & 0 & \frac{1}{L} \end{bmatrix} \mathbf{u} \equiv \mathbf{Cu} \tag{13}$$

where the row vectors $\mathbf{B}$ and $\mathbf{C}$ are defined. That means the strain in Eq. (2) is written

$$\varepsilon = \mathbf{Bu} + \frac{1}{2} \cdot (\mathbf{Cu})^T\, \mathbf{Cu} = \mathbf{Bu} + \frac{1}{2} \cdot \mathbf{u}^T \mathbf{C}^T \mathbf{Cu} \tag{14}$$

That notation also means that the virtual strain in Eq. (9) reads

$$\delta\varepsilon = \delta u' + w' \cdot \delta w' = \mathbf{B}\delta\mathbf{u} + \mathbf{u}^T \mathbf{C}^T \mathbf{C}\delta\mathbf{u} \tag{15}$$

Substitution of Eqs. (4), (14), and (15) into the internal work expression in Eq. (8) yields

$$EA \cdot \int_0^L \left(\mathbf{Bu} + \frac{1}{2} \cdot \mathbf{u}^T \mathbf{C}^T \mathbf{Cu} \right) \left(\mathbf{B}\delta\mathbf{u} + \mathbf{u}^T \mathbf{C}^T \mathbf{C}\delta\mathbf{u} \right) dx = \int_0^L \mathbf{p}^T \left(\mathbf{N}\delta\mathbf{u} \right) dx \tag{16}$$

Rearranging like we always do in the derivation of finite elements, i.e., changing the order of multiplication of the scalars within the parenthesis, taking the transpose of a scalar, and extracting $\delta\mathbf{u}$, assuming arbitrary virtual displacements yields

$$EA \cdot \int_0^L \left(\mathbf{B}^T + \mathbf{C}^T\mathbf{C}\mathbf{u}\right)\left(\mathbf{B}\mathbf{u} + \frac{1}{2}\cdot\mathbf{u}^T\mathbf{C}^T\mathbf{C}\mathbf{u}\right)dx = \int_0^L \mathbf{N}^T\mathbf{p}\,dx \tag{17}$$

EA multiplied by the second parenthesis is the axial force in the element because

$$N = A\cdot\sigma = EA\cdot\varepsilon = EA\cdot\left(\mathbf{B}\mathbf{u} + \frac{1}{2}\cdot\mathbf{u}^T\mathbf{C}^T\mathbf{C}\mathbf{u}\right) \tag{18}$$

That means Eq. (17) can be written

$$\overbrace{\int_0^L \mathbf{B}^T\cdot N\,dx}^{\text{Linear term}} + \overbrace{\int_0^L \mathbf{C}^T\mathbf{C}\mathbf{u}\cdot N\,dx}^{\text{Geometric nonlinearity}} = \int_0^L \mathbf{N}^T\mathbf{p}\,dx \tag{19}$$

which, following the notation established on this website, symbolically reads $\tilde{\mathbf{F}} = \mathbf{F}$. That is a useful expression for nonlinear structural analysis, where the axial force, *N*, is determined from the trial displacements so that the internal resisting forces, $\tilde{\mathbf{F}}$, can be calculated. Substitution of the expressions for **B** and **C** from Eqs. (12) and (13) gives

$$\begin{aligned}\tilde{\mathbf{F}} &= \begin{Bmatrix}-1\\0\\1\\0\end{Bmatrix}N + \begin{bmatrix}0&0&0&0\\0&\frac{1}{L}&0&-\frac{1}{L}\\0&0&0&0\\0&-\frac{1}{L}&0&\frac{1}{L}\end{bmatrix}\begin{Bmatrix}u_1\\u_2\\u_3\\u_4\end{Bmatrix}N\\ &= \begin{Bmatrix}-1\\0\\1\\0\end{Bmatrix}N + \begin{Bmatrix}0\\-\left(\frac{u_4-u_2}{L}\right)\\0\\\left(\frac{u_4-u_2}{L}\right)\end{Bmatrix}N = \begin{Bmatrix}-1\\-\theta\\1\\\theta\end{Bmatrix}N\end{aligned} \tag{20}$$

where θ is introduced to denote the counter-clockwise chord rotation of the element. Eq. (20) represents equilibrium in the displaced shape marked with the red colour in Figure 1, relative to the previously committed shape marked with blue. Furthermore, Eq. (20) defines the transformation from the Basic to the Local configuration; Eq. (20) shows that the force along the four Local degrees of freedom is equal to a vector times the Basic axial force, *N*. That means the vector in the last equality in Eq. (20) is the transpose of $\mathbf{T}_{bl}$. That is reflected in the implementation as Element 3 in G2, posted on this website. Notice that θ in $\mathbf{T}_{bl}$ is the rotation from the blue reference configuration in Figure 1 to the current red configuration. It is useful for the top-level solution algorithm to also have the tangent stiffness matrix:

$$\mathbf{K}_T = \frac{\partial\tilde{\mathbf{F}}}{\partial\mathbf{u}} = \int_0^L \mathbf{B}^T\cdot\frac{\partial N}{\partial\mathbf{u}}dx + \int_0^L \mathbf{C}^T\mathbf{C}\mathbf{u}\cdot\frac{\partial N}{\partial\mathbf{u}}dx + \int_0^L \mathbf{C}^T\mathbf{C}\cdot\mathbf{I}\cdot N\,dx \tag{21}$$

where the product rule of differentiation is employed, and where, in light of Eq. (18):

$$\frac{\partial N}{\partial\mathbf{u}} = EA\cdot\left(\mathbf{B} + \mathbf{u}^T\mathbf{C}^T\mathbf{C}\right) \tag{22}$$

That means the tangent stiffness is

$$\mathbf{K}_T = \overbrace{\int_0^L EA \cdot \mathbf{B}^T \mathbf{B}\, dx}^{\mathbf{K}_o} + \int_0^L EA \cdot \mathbf{B}^T \mathbf{u}^T \mathbf{C}^T \mathbf{C}\, dx + \int_0^L EA \cdot \mathbf{C}^T \mathbf{C} \mathbf{u} \mathbf{B}\, dx + \int_0^L EA \cdot \mathbf{C}^T \mathbf{C} \mathbf{u} \mathbf{u}^T \mathbf{C}^T \mathbf{C}\, dx + \underbrace{\int_0^L \mathbf{C}^T \mathbf{C} \cdot N\, dx}_{\mathbf{K}^{\mathrm{G}}} \tag{23}$$

where the elastic and geometric stiffness matrices are identified. As explained on Page 68 of the second edition of the Crisfield book, the tangent stiffness itself is not a function of **u**, implying that only $\mathbf{K}_{\mathrm{o}}$ and $\mathbf{K}^{\mathrm{G}}$ are kept in the implementation of this element as Element 3 in the G2 code posted on this website. Also, in the implementation in Element 3, the elastic portion of the stiffness matrix is simply calculated as $\mathbf{K}_{\mathrm{o}}=\mathbf{T}_{\mathrm{bl}}^{\mathrm{T}}\mathbf{K}_{\mathrm{b}}\mathbf{T}_{\mathrm{bl}}$, with defined in the discussion after Eq. (20). Notice that $\mathbf{K}_{\mathrm{o}}$ and $\mathbf{K}^{\mathrm{G}}$ are stiffness matrices in the Local element configuration, marked with blue in Figure 1. In order to obtain the global stiffness matrix that is returned from the element, the transformation

$$\mathbf{K}_{\mathrm{g}} = \mathbf{T}_{\mathrm{lg}}^{\mathrm{T}}(\mathbf{K}_{\mathrm{o}} + \mathbf{K}^{\mathrm{G}})\mathbf{T}_{\mathrm{lg}} \tag{24}$$

is conducted with the standard local-to-global transformation

$$\mathbf{T}_{\mathrm{lg}} = \begin{bmatrix} c_x & c_y & 0 & 0 \\ -c_y & c_x & 0 & 0 \\ 0 & 0 & c_x & c_y \\ 0 & 0 & -c_y & c_x \end{bmatrix} \tag{25}$$

where the direction cosines are $c_x=\Delta\mathbf{x}/L=\cos(\theta)$ and $c_y=\Delta\mathbf{z}/L=\sin(\theta)$ with $\Delta\mathbf{x}$ and $\Delta\mathbf{z}$ being the difference in *x*- and *z*-coordinates of the element ends, including both the original undeformed configuration and also the displacements up to the previously committed state. In other words, θ is here the angle between the *x*-axis and the blue line in Figure 1. The remaining rotation from the blue line to the red line was addressed in the $\mathbf{T}_{\mathrm{bl}}$ transformation described earlier.

References

Crisfield (1991). "Nonlinear Finite Element Analysis of Solids and Structures." Volume 1. First Edition. Wiley.

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