System Reliability

System reliability analysis addresses problems where the failure event is defined by the joint state of more than one limit-state function. As a conceptual example, consider a structural element with two failure modes, each associated with one limit-state function. The individual limit-state functions are called "component reliability problems" and are addressed by, for instance, FORM analysis. However, if failure occurs only if both limit-state functions have a negative realization, then it is a "system reliability problem." Specifically, in this conceptual example, it is a parallel system problem, as described below.

Formulation of System Reliability Problems

For reference, the component reliability problem reads

$$p_f = P\big(g(\mathbf{x}) \le 0\big) \tag{1}$$

In contrast, a series system reliability problem is characterized by failure occurring if any of several limit-state functions have negative realizations:

$$p_f = P\left(\bigcup_{k=1}^{K}\big(g_k(\mathbf{x}) \le 0\big)\right) \tag{2}$$

where K is the number of limit-state functions. Another special case of a system reliability problem is the parallel system, in which failure occurs only when all the limit-state functions have negative realizations:

$$p_f = P\left(\bigcap_{k=1}^{K}\big(g_k(\mathbf{x}) \le 0\big)\right) \tag{3}$$

Neither the series system nor the parallel system describes the general system reliability problem, but both concepts are involved. The general system reliability problem is usually formulated as a series system of parallel systems:

$$p_f = P\left(\bigcup_{m=1}^{M}\bigcap_{j\in c_m}\big(g_j(\mathbf{x}) \le 0\big)\right) \tag{4}$$

where M is the number of sub-parallel systems and c_m is the "cut set" that contains the indices of the limit-state functions that form sub-parallel system m. The logical operator $j \in c_m$ in Eq. (4) means that only limit-state functions within cut set m is included. The cut set formulation in Eq. (4) can be replaced by the equally general "link set" formulation, in which the problem is defined as a parallel system of sub series systems. However, this is uncommon and less intuitive in the reliability analyses addressed here.

Sampling

In this document it will soon become apparent that the calculation of the exact failure probability of a system is hard, at best. Although sampling analysis is often computationally expensive it is appealingly straightforward to implement for system reliability problems. For each sample is it checked whether the any of the limit-states are violated and if this caused system failure. Depending on the result the indicator function explained in the Sampling notes on this website is set to zero or unity, and another sample is generated. Hence, sampling is always a straightforward alternative to obtain an estimate of the system failure probability, albeit often a computationally costly one.

FORM Analysis

If FORM analysis is conducted for each individual component reliability problem then the system failure probability can be obtained for certain series and parallel systems. General system reliability problems are addressed in later sections.

Parallel Systems

The parallel system problem in Eq. (3) is equivalently written in the standard normal space as

$$p_f = P\left(\bigcap_{k=1}^{K}\left(G_k(\mathbf{y}) \leq 0\right)\right) \tag{5}$$

where G=limit-state function and $\mathbf{y}$=vector of standard normal random variables. Linearizing each limit-state function by FORM yields

$$p_f = P\left(\bigcap_{k=1}^{K}\left(\beta_k - \boldsymbol{\alpha}_k^T\mathbf{y} \leq 0\right)\right) \tag{6}$$

where β_k and $\boldsymbol{\alpha}_k$ are the reliability index and alpha-vector for each limit-state function. For each limit-state function, the random variable

$$z_k = \boldsymbol{\alpha}_k^T\mathbf{y} \tag{7}$$

is now defined. According to the joint uncorrelated standard normal distribution for $\mathbf{y}$ the variables z_k have zero mean. Also, because the alpha-vectors are normalized the z_k variables have unit variances. Conversely, the covariance between z_i and z_j is

$$\text{Cov}\left[\boldsymbol{\alpha}_i^T\mathbf{y}, \boldsymbol{\alpha}_j^T\mathbf{y}\right] = \boldsymbol{\alpha}_i^T\boldsymbol{\Sigma}_{yy}\boldsymbol{\alpha}_j = \boldsymbol{\alpha}_i^T\boldsymbol{\alpha}_j \tag{8}$$

i.e., the dot product of the two alpha-vectors. This covariance is equal to the correlation because the z_k variables have unit standard deviation. Substitution of Eq. (7) into Eq. (6) yields the system failure probability

$$p_f = P\left(\bigcap_{k=1}^{K}\left(\beta_k \leq z_k\right)\right) \tag{9}$$

Because of the symmetry of the normal distribution that governs the variables z_k, Eq. (9) can be rewritten in the form

$$p_f = P\left(\bigcap_{k=1}^{K}(z_k < -\beta_k)\right) \tag{10}$$

which expresses the *k*-dimensional correlated standard normal CDF; hence, the system failure probability can be written (Hohenbichler and Rackwitz 1983; Der Kiureghian 2005)

$$p_f = \Phi_K(-\boldsymbol{\beta}, \mathbf{R}) \tag{11}$$

where $\boldsymbol{\beta}$ is the vector of reliability indices for all the limit-state functions and $\mathbf{R}$ is the correlation matrix for the random variables z_k. The correlation matrix contains the covariances in Eq. (8) because the standard deviations are equal to one. Evaluation of the joint normal CDF is addressed shortly.

Series Systems

The series system problem in Eq. (2) is equivalently written in the standard normal space as

$$p_f = P\left(\bigcup_{k=1}^{K}(G_k(\mathbf{y}) \leq 0)\right) \tag{12}$$

and repeating the FORM linearization from above yields

$$p_f = P\left(\bigcup_{k=1}^{K}(\beta_k - \boldsymbol{\alpha}_k^T\mathbf{y} \leq 0)\right) \tag{13}$$

again defining the variables

$$z_k = \boldsymbol{\alpha}_k^T\mathbf{y} \tag{14}$$

which have zero mean and unit variances, with the covariance, here equal to the correlation, between z_i and z_j written

$$\text{Cov}\left[\boldsymbol{\alpha}_i^T\mathbf{y}, \boldsymbol{\alpha}_j^T\mathbf{y}\right] = \boldsymbol{\alpha}_i^T\boldsymbol{\Sigma}_{yy}\boldsymbol{\alpha}_j = \boldsymbol{\alpha}_i^T\boldsymbol{\alpha}_j \tag{15}$$

Substitution of Eq. (14) into Eq. (13) yields the system failure probability

$$p_f = P\left(\bigcup_{k=1}^{K}(\beta_k \leq z_k)\right) \tag{16}$$

The inclusion-exclusion rule is unappealing for evaluating that probability; to arrive at an expression involving the intersection operator rather than the union operator the complementary probability rule is first invoked:

$$p_f = 1 - P\left(\overline{\bigcup_{k=1}^{K}(\beta_k \leq z_k)}\right) \tag{17}$$

followed by the application of one of de Morgan's rules:

$$p_f = 1 - P\left(\bigcap_{k=1}^{K} \overline{(\beta_k \le z_k)}\right) = 1 - P\left(\bigcap_{k=1}^{K} (z_k < \beta_k)\right) \tag{18}$$

Because z_k are correlated standard normal random variables the k-dimensional joint standard normal CDF is identified; hence, the probability reads (Hohenbichler and Rackwitz 1983; Der Kiureghian 2005)

$$p_f = 1 - \Phi_K(\boldsymbol{\beta}, \mathbf{R}) \tag{19}$$

where again $\boldsymbol{\beta}$ is the vector of reliability indices for all the limit-state functions and $\mathbf{R}$ is the correlation matrix for the random variables z_k.

Evaluating the Multivariate Standard Normal CDF with Correlation

Eqs. (11) and (19) look good on paper but it is unfortunately difficult to evaluate the k-dimensional joint standard normal CDF (Ambartzumian et al. 1998). However, for k=2 matters are somewhat simpler and the sought expression is

$$\begin{gathered}\Phi_2(\boldsymbol{\beta}, \mathbf{R}) = \Phi(\beta_1)\cdot\Phi(\beta_2) + \int_0^{\rho_{12}} \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left[-\frac{\beta_1^2 + \beta_2^2 - 2\cdot\rho\cdot\beta_1\cdot\beta_2}{2\cdot(1-\rho^2)}\right] d\rho \\ \Updownarrow \\ \Phi_2(-\boldsymbol{\beta}, \mathbf{R}) = \Phi(-\beta_1)\cdot\Phi(-\beta_2) + \int_0^{\rho_{12}} \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left[-\frac{\beta_1^2 + \beta_2^2 - 2\cdot\rho\cdot\beta_1\cdot\beta_2}{2\cdot(1-\rho^2)}\right] d\rho\end{gathered} \tag{20}$$

where the last term is the integral of the bivariate standard normal PDF with correlation, with correlation between the two limit-states explained earlier as

$$\rho_{12} = \boldsymbol{\alpha}_1^T \boldsymbol{\alpha}_2 \tag{21}$$

It is noted that this correlation coefficient must exceed, say, 0.5 before the intersection probability, i.e., the integral in Eq. (20) becomes significant. Given Eq. (20) it is possible, in general, to evaluate parallel and series system problems with up to two components. Larger parallel and series systems can be analyzed if the limit-states are uncorrelated, but this is still a rather disappointing situation. However, Eq. (20) does open up an approximate analysis venue; namely, the use of bounds, which is addressed next.

System Reliability Bounds

The difficulties associated with evaluating the system failure probability even for series and parallel systems have led to the exploration of probability bounds. That approach is here explained for series systems, which also addresses parallel systems in the sense that the latter can be transformed into series systems using de Morgan's rules. In the following, component m, i.e., each failure mode m of the series system is denoted c_m. This failure mode can either be a single limit-state function or a cut set. If c_m represents a cut set then remember, as explained later in this document, that it can be challenging to define it and calculate $P(c_m)$ and the bi-modal probabilities $P(c_i c_j)$, which will appear shortly. The bounds on the series system failure probability are derived by first considering the inclusion-exclusion rule

$$p_f = P\left(\bigcup_{m=1}^{M} c_m\right) = \sum_{m=1}^{M} P(c_m) - \sum_{\substack{m=1 \\ i<m}}^{M} P(c_m c_i) + \sum_{\substack{m=1 \\ j<i<m}}^{M} P(c_m c_i c_j) - \cdots \qquad (22)$$

where c_m denotes the event that limit-state number m is violated, or more generally that all limit-states in cut set number m is in the failure state. Introducing the shorthand notation p_m=P(c_m) the so-called unimodal bounds are formulated by retaining only the first term in Eq. (22):

$$\max(p_m) \quad \leq \quad p_f \quad \leq \quad \min\left(1, \sum_{m=1}^{M} p_m\right) \qquad (23)$$

Uni-modal bounds make for a rough approximation. A better option, made available through Eq. (20), are bi-modal bounds, which represent an extension where also the two-component intersection probabilities are kept from Eq. (22). Using the shorthand notation p_{ij}=P($c_i c_j$) the bounds read (Ditlevsen 1979)

$$p_1 + \sum_{m=2}^{M} \max\left(0, p_m - \sum_{j=1}^{m-1} p_{mj}\right) \quad \leq \quad p_f \quad \leq \quad \mathrm{p}_1 + \sum_{m=2}^{M}\left(p_m - \max_{j<m}(p_{mj})\right) \qquad (24)$$

where the bi-modal probabilities p_{ij} are evaluated according to Eq. (20). To make these bi-modal bounds more accessible they are here illustrated for up to four components:

$$\begin{array}{l} p_1 + \max\left(0, p_2 - p_{21}\right) \\ +\max\left(0, p_3 - (p_{31} + p_{32})\right) \\ +\max\left(0, p_4 - (p_{41} + p_{42} + p_{43})\right) \end{array} \quad \leq \quad p_f \quad \leq \quad \begin{array}{l} p_1 + p_2 - p_{21} \\ +p_3 - \max(p_{31}, p_{32}) \\ +p_4 - \max(p_{41}, p_{42}, p_{43}) \end{array} \qquad (25)$$

The bounds will vary somewhat depending on the ordering of the limit-state functions; hence, all combinations should be checked to ensure that the bounds are not falsely narrow.

Identification of Cut Sets for General Systems

Unless the problem at hand is a "pure" series or parallel system, it is an important task to identify the cut sets of the problem. To repeat, a cut set is a collection of limit-states whose joint failure causes failure of the system. Three concepts are helpful when identifying the cut sets of a general system reliability problem:

- Reliability block diagrams
- Minimum cut sets
- Disjoint cut sets

These concepts are described in the following using a simple example.

Reliability Block Diagrams

A reliability block diagram (RBD), such as the illustration in Figure 1, can informally be regarded as a road map between two cities. In this view, the individual limit-state functions, numbered 1 through 4 in the figure, are informally seen as bridges that can in the safe or failed state. If certain bridges are closed, i.e., if certain limit-states are in the failure state then it is impossible to get from one city to the other, i.e., from one of the solid black circles to the other. For example, in Figure 1 the system fails if limit-state 3 and 4 fail. This means that components 3 and 4 constitutes a cut set. The roadway interpretation of an RBD helps when identifying cut sets; the failure of any set of components that would close the connection between the two cities is a cut set.

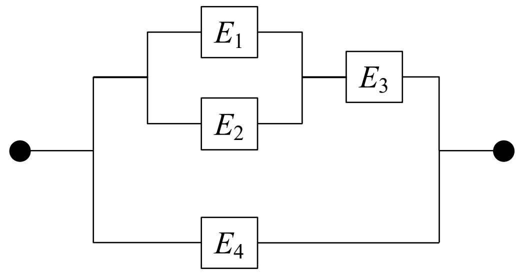

Figure 1: Reliability block diagram.

Minimum Cut Sets

The first task in solving a general system reliability problem is to identify all the cut sets. The system visualized in the RBD in Figure 1 has two cut sets:

$$\begin{aligned} c_1 &= \{E_3, E_4\} \\ c_2 &= \{E_1, E_2, E_4\} \end{aligned} \tag{26}$$

Using the informal parlance of the previous subsection, this is because the failure of "bridge" 3 and 4, or the failure of bridge 1, 2, and 4 will close the connection between the two cities marked by solid black dots. In short, identify cut sets by identifying all combinations of component failures that would cause system failure. Usually these are immediately "minimum" cut sets; namely, cut sets that contain the minimum number of components. By definition, a minimum cut set is formulated so that, if any component is removed from the cut set, then it ceases to be a cut set.

Disjoint Cut Sets

To understand the meaning and application of "disjoint cut sets" it is useful to see how the system failure probability is calculated with minimum cut sets, i.e., using the inclusion-exclusion rule:

$$p_f = P\left(\bigcup_{m=1}^{M} c_m\right) = \sum_{m=1}^{M} P(c_m) - \sum_{\substack{m=1 \\ i<m}}^{M} P(c_m c_i) + \sum_{\substack{m=1 \\ j<i<m}}^{M} P(c_m c_i c_j) - \cdots \tag{27}$$

While Eq. (27) may seem conceptually straightforward, it is actually tricky to ensure that the correct events are subtracted and added in the intersection probabilities. This problem motivates the formulation of disjoint cut sets, i.e., cut sets that are mutually exclusive. This means that the intersection probabilities in Eq. (27) are zero, and the failure probability for the system is

$$p_f = P\left(\bigcup_{m=1}^{M} c_m\right) = \sum_{m=1}^{M} P(c_m) \tag{28}$$

Although Eq. (28) simplified greatly from Eq. (27) an important challenge remains; namely, to define proper disjoint cut sets. For the simple example in Figure 1 it is tempting to simply revise Eq. (26) to

$$\begin{aligned} c_1 &= \{\bar{E}_1, E_3, E_4\} \\ c_2 &= \{E_1, E_2, E_4\} \end{aligned} \tag{29}$$

because these cannot occur simultaneously. However, that simplistic approach is fraught with peril. Only a more careful inspection of a Venn diagram, or even better, an enumeration of all possible mutually exclusive and collectively exhaustive system states, can tell whether we are including the desired events in the definition of system failure. In order to explain that, and also in preparation for the next section, consider the Venn diagram in Figure 2. It shows the case of two components. Each component can either be in a safe or failed state. For such two-state components, the number of possible disjoint system states is 2^N, where N is the number of components. Figure 2 identifies by symbols e_i the four disjoint system states for the case of two components.

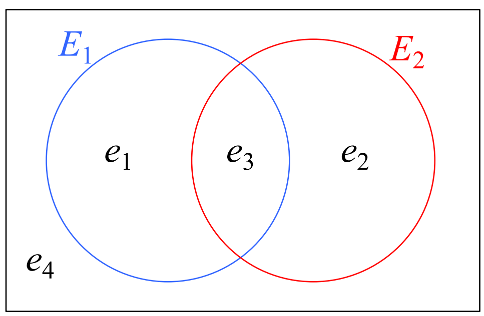

Figure 2: Venn diagram showing distinct system states for two components ($2^N=2^2=4$).

Figure 3 identifies all possible disjoint system states for the case of three two-state components. With only three components, it is still a straightforward task to enumerate all disjoint system states. For example, we easily identify $e_5 = \{E_1, \bar{E}_2, E_3\}$, $e_8 = \{\bar{E}_1, \bar{E}_2, \bar{E}_3\}$, and so forth.

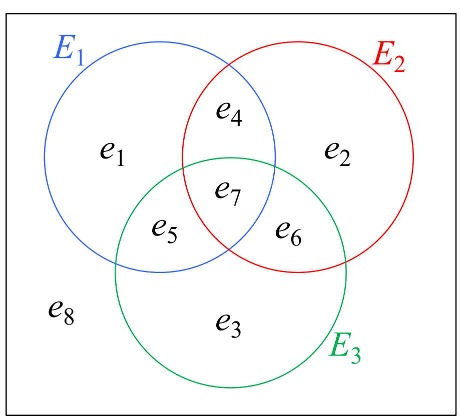

Figure 3: Venn diagram showing distinct system states for three components ($2^N=2^3=8$).

Figure 4 shows the Venn diagram for four components. Already with this low number of components, it is more challenging to identify all disjoint system states, which there are now sixteen of. The system states corresponding to the two cut sets from Eq. (26) are marked with boldface. That reveals that the attempt to formulate disjoint cut sets in Eq. Eq. (29) misses a system state that causes failure. Specifically, the first cut set in Eq. (29) encompasses e_{10} and e_{13} in Figure 4 and the second cut set encompasses e_{12} and e_{15}. That leaves e_{14} unaccounted for, implying a failed attempt in Eq. (29) to formulate proper disjoint cut set. A more systematic approach is explained next.

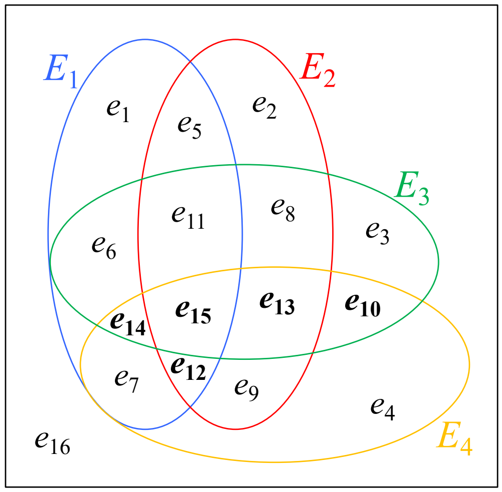

Figure 4: Venn diagram showing distinct system states for four components ($2^N=2^4=16$).

Matrix Formulation

During the first years of this millennium, I studied together with Junho Song, Paolo Gardoni, and Johannes Royset in an office space created by our supervisor Professor Der Kiureghian. My study mates are geniuses in their own right, now holding prestigious professor positions at top universities. However, Professor Song possesses a unique combination of a kind and calm personality, exceptional academic creativity, and a mathematical rigour only matched by our supervisor. It is Junho Song who is the person behind the ideas outlined in this section. For additional details and clarity, read Professor Song's papers and the system reliability chapter in the 2022 textbook by Professor Der Kiureghian. As an introduction, consider a selection vector, **a**, with the same dimension as the number of mutually exclusive and collectively exhaustive events, i.e., 2^N. The purpose of **a** is to pick from the mutually exclusive and collectively exhaustive events the ones that defines system failure. For the example addressed earlier, the boldface events in Figure 4 suggests that the selection vector in that case is

$$\mathbf{a}^{\mathrm{T}} = \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 \end{matrix} \tag{30}$$

For a moment, suppose the probability of occurrence of all the mutually exclusive and collectively exhaustive events are available. Denote them by $p_i=\mathrm{P}(e_i)$ and collect them in the vector **p**. The failure probability for the system is then the dot product

$$p_f = \mathbf{a}^{\mathrm{T}}\mathbf{p} \tag{31}$$

Because several values in **p** are typically unavilable, Professor Song proposed the use of Eq. (31) as the objective function in a linear programming formulation. That facilitates the formulation of equality and inequality constraints for the probabilities in **p** and the calculation of system probability bounds by linear programming. However, let us return to the formulation of **a**. Without a Venn diagram, such as that in Figure 4, it can be awkward and error-prone to establish that selection vector. Professor Song came up with clever way to establish **a**, which acknowledges that a cut set formulation like Eq. (26) is the starting point. The cut sets spells out what is system failure, in terms of the components, E_j. Denoting by $\mathbf{a}_j$ the 2^N-dimensional selection vector for each component, the selection vectors for the previously considered example are

$$\begin{array}{rcccccccccccccccc}
 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 \\
\mathbf{a}_1^{\mathrm{T}} = & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\
\mathbf{a}_2^{\mathrm{T}} = & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 \\
\mathbf{a}_3^{\mathrm{T}} = & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 \\
\mathbf{a}_4^{\mathrm{T}} = & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0
\end{array} \tag{32}$$

In order to use those vectors in the cut set formulation of systems, recall that each cut set is a parallel system, and that the cut sets then form a series system. From Eq. (26), the first cut set consists of components E_3 and E_4. The element-wise Hadamard multiplication of $\mathbf{a}_3$ and $\mathbf{a}_4$ gives the selection vector for that cut set:

$$\begin{array}{rcccccccccccccccc}
 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 \\
\mathbf{a}_3^{\mathrm{T}} = & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 \\
\mathbf{a}_4^{\mathrm{T}} = & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 \\
\mathbf{a}_3^{\mathrm{T}} * \mathbf{a}_4^{\mathrm{T}} = & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0
\end{array} \tag{33}$$

We observe that the result is a selection vector that picks e_{10}, e_{13}, e_{14}, and e_{15} from Figure 4. From Eq. (26), the second cut set consists of components E_1, E_2, and E_4. The element-wise Hadamard multiplication of $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_4$ gives the selection vector for that cut set:

$$\begin{array}{rcccccccccccccccc}
 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 \\
\mathbf{a}_1^{\mathrm{T}} = & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\
\mathbf{a}_2^{\mathrm{T}} = & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 \\
\mathbf{a}_4^{\mathrm{T}} = & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 \\
\mathbf{a}_1^{\mathrm{T}} * \mathbf{a}_2^{\mathrm{T}} * \mathbf{a}_4^{\mathrm{T}} = & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0
\end{array} \tag{34}$$

We observe that the result is a selection vector that picks e_{12} and e_{15} from Figure 4. At this stage, it is comforting to see that the results in Eqs. (33) and (34) include all the events marked with boldface in Figure 4. However, we must now merge those results to obtain the final selection vector. That is done by recognizing that the cut sets formulated in Eq. (26) and reformulated in Eqs. (33) and (34) form a series system. Instead of the multiplication above, it would *not* make sense to engage in summation of the vectors obtained in Eqs. (33) and (34). That would give values greater than unity in the selection vector. Rather, use the following sketch of the probability of the complement in conjunction with one of de Morgan's rules:

$$\begin{aligned} P(c_1 \cup c_2) = 1 - P(\overline{c_1 \cup c_2}) = 1 - P(\overline{c_1} \cap \overline{c_2}) = 1 - P(\overline{c_1}) \cdot P(\overline{c_2}) \\ = 1 - \big(1 - P(c_1)\big) \cdot \big(1 - P(c_1)\big) \end{aligned} \quad (35)$$

to underpin the following merger of the selection vectors that came out of Eqs. (33) and (34) in order to obtain the final selection vector:

$$\mathbf{a} = \mathbf{1} - (\mathbf{1} - \mathbf{a}_3 * \mathbf{a}_4) * (\mathbf{1} - \mathbf{a}_1 * \mathbf{a}_2 * \mathbf{a}_4) \quad (36)$$

where **1** is a 2^N-dimensional vecot with ones and the symbol * still means Hadamard multiplication. Evaluating Eq. (36) for the considered example gives a vector **a** that is identical to Eq. (30), as expected.

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