A short course on

Nonlinear Finite Element Analysis

This video:

Nonlinear Frame Elements

Terje's Toolbox is freely available at terje.civil.ubc.ca

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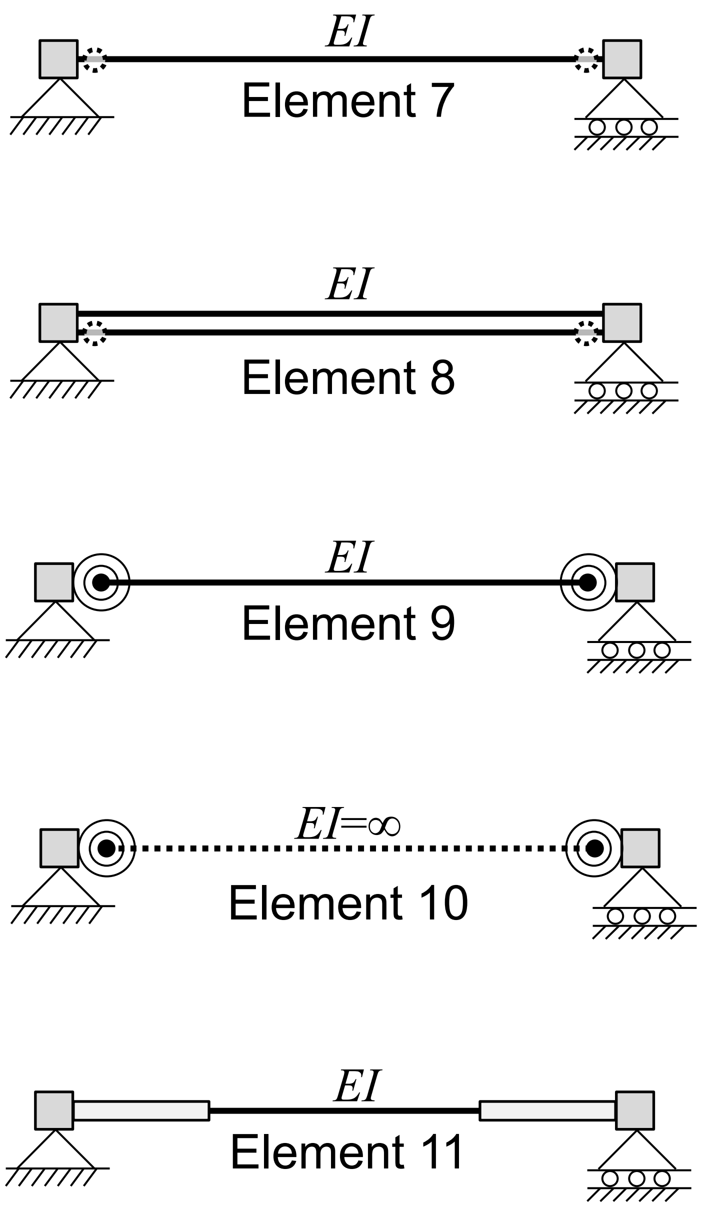
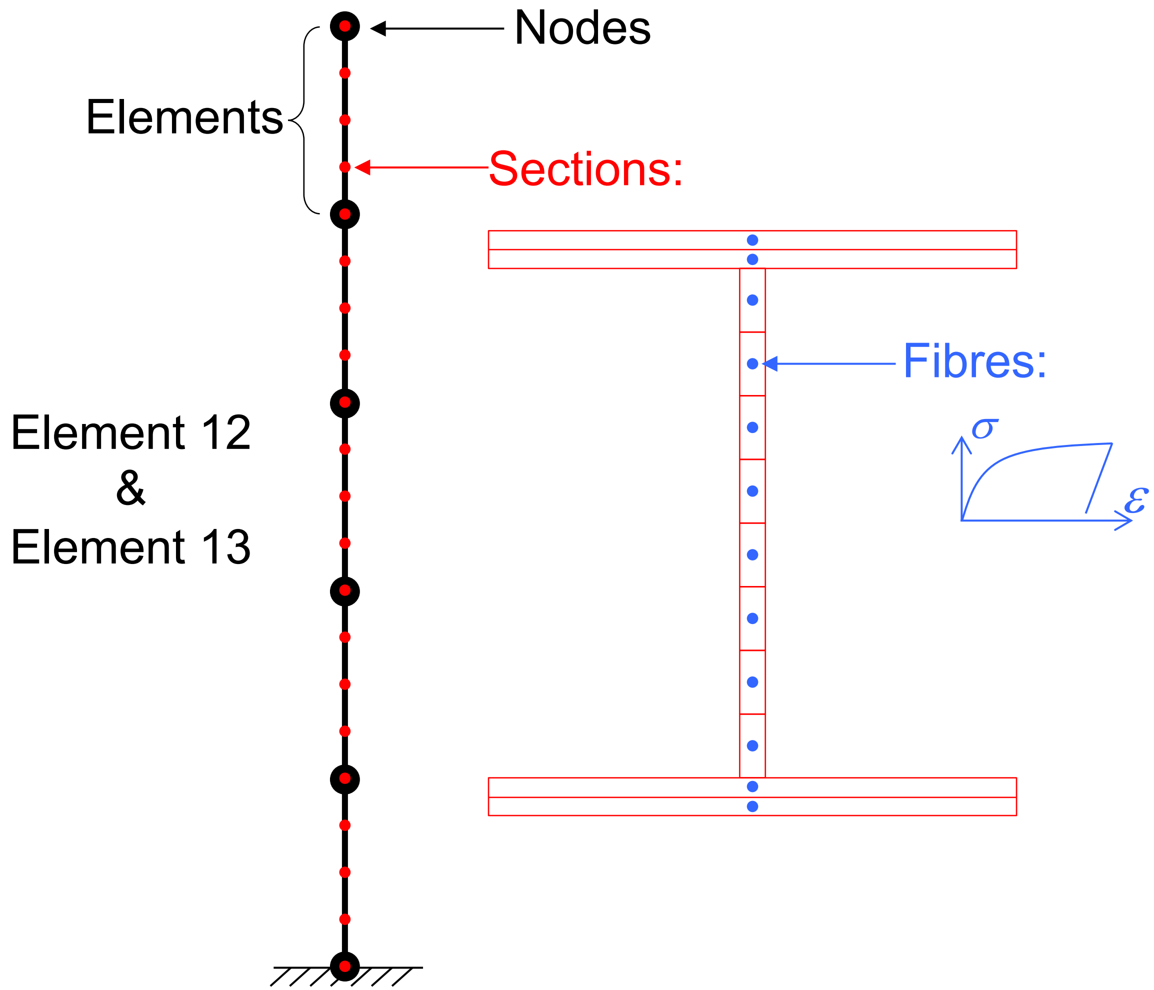
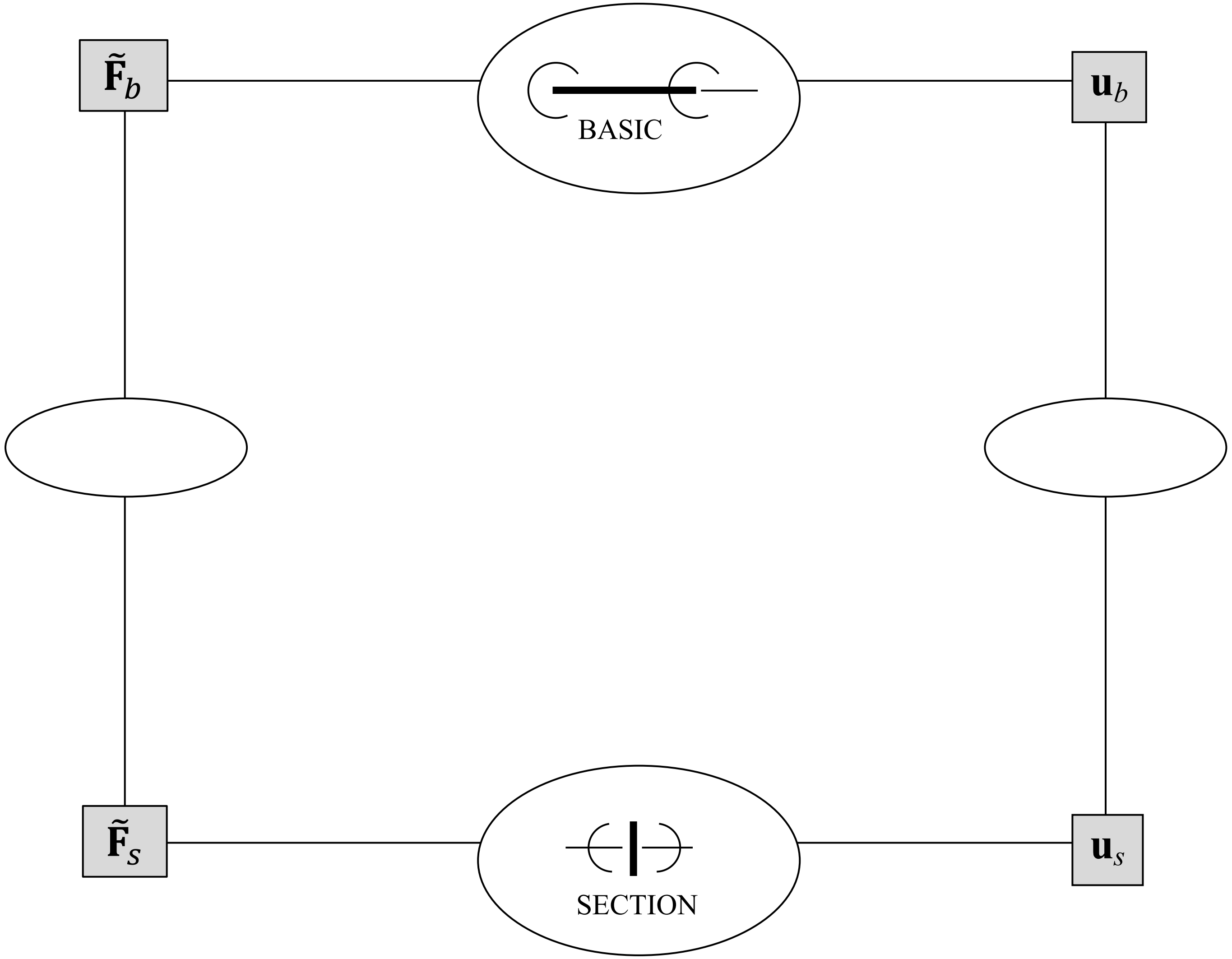
Element Types

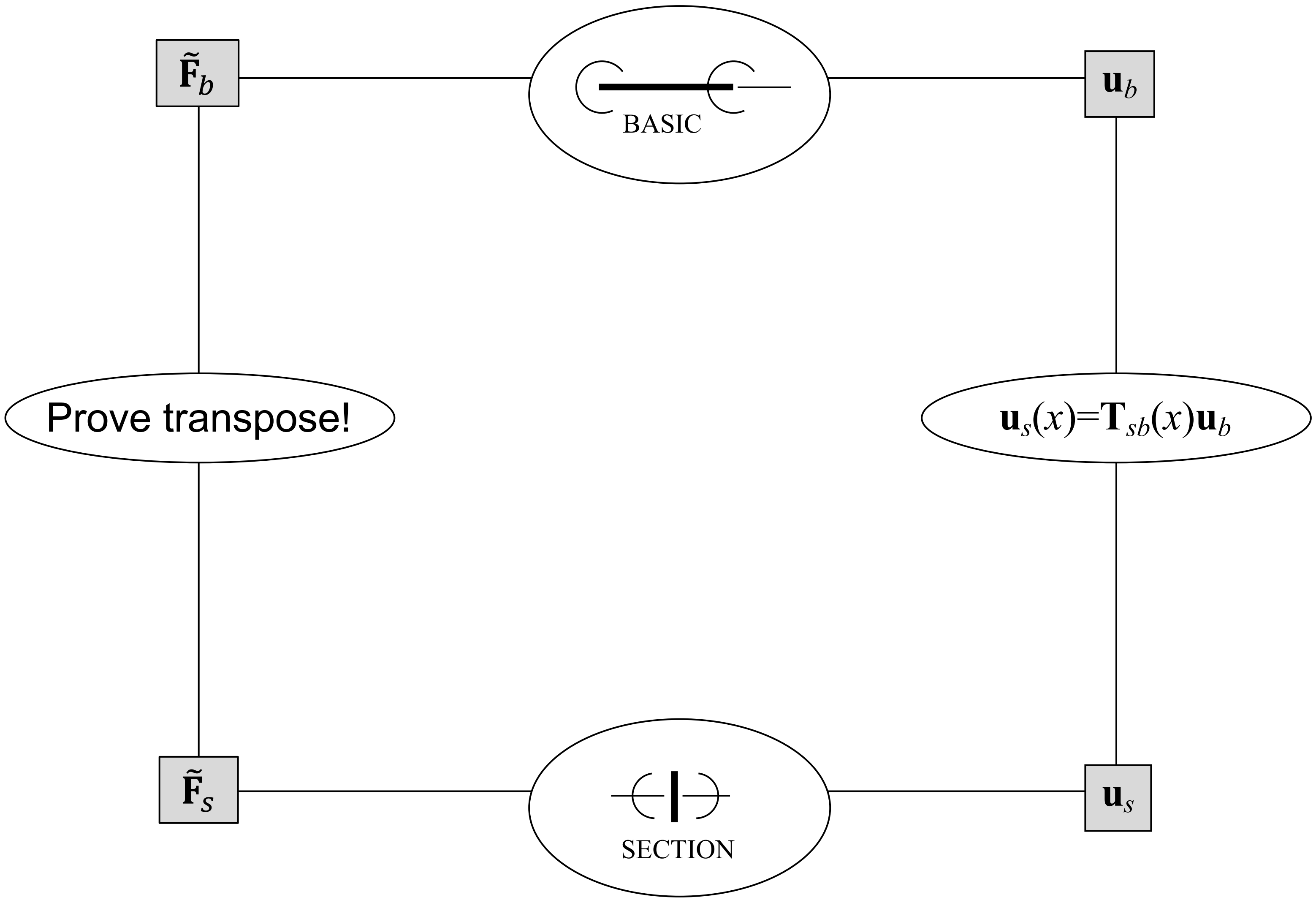




Linear Exercise



Displacement Interpolation



Stiffness Matrix

Principle of virtual displacements: $\delta\mathbf{u}_b^{\mathrm{T}}\mathbf{F}_b = \int_0^L \delta\mathbf{u}_s^{\mathrm{T}}\,\mathbf{F}_s dx$

Material law: $\delta\mathbf{u}_b^{\mathrm{T}}\mathbf{F}_b = \int_0^L \delta\mathbf{u}_s^{\mathrm{T}}\,(\mathbf{K}_s\mathbf{u}_s)dx$

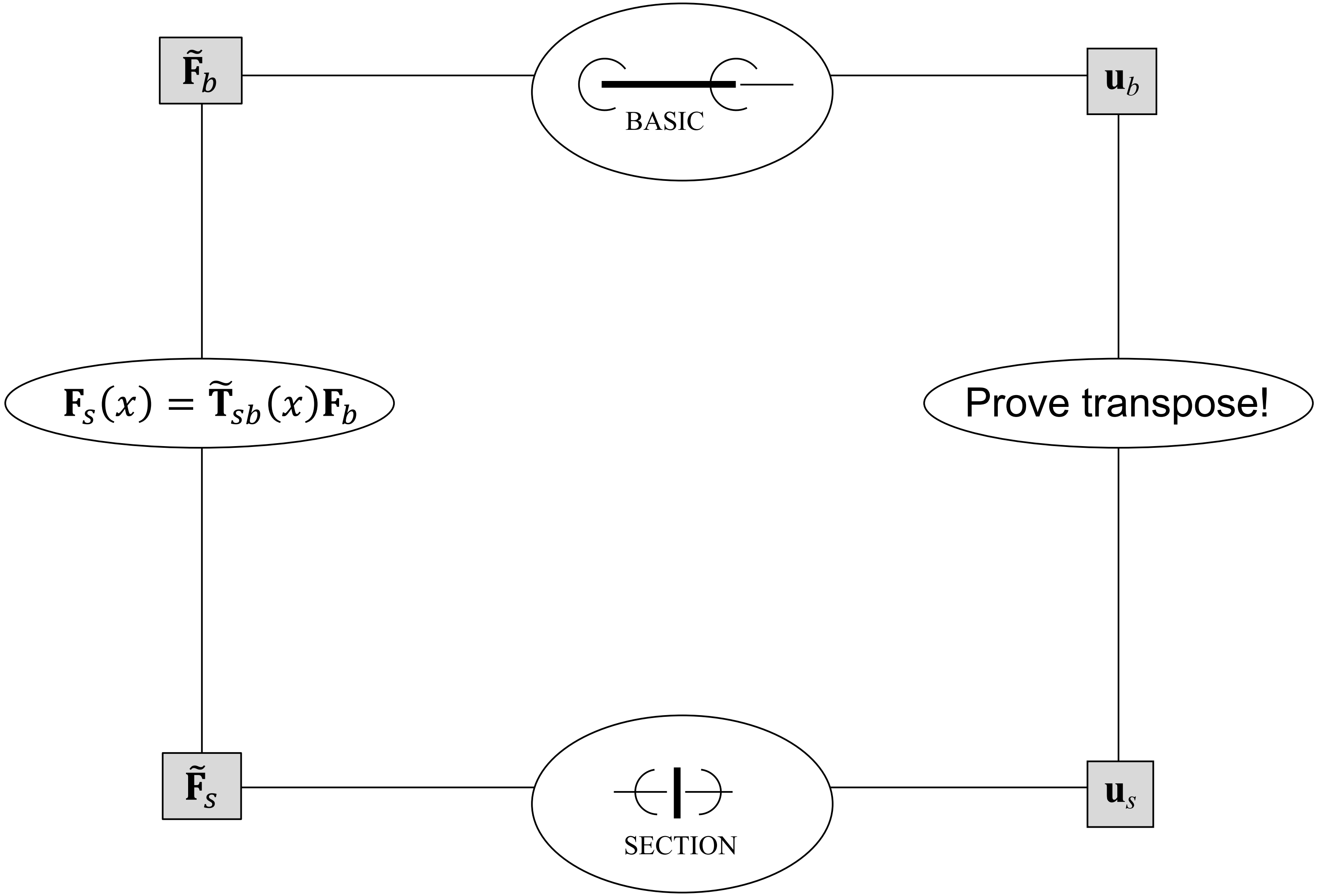
Kinematic compatibility: $\delta\mathbf{u}_b^{\mathrm{T}}\mathbf{F}_b = \int_0^L \left(\delta\mathbf{u}_{\mathrm{b}}^{\mathrm{T}}\mathbf{T}_{sb}^{\mathrm{T}}\right)\mathbf{K}_s(\mathbf{T}_{sb}\mathbf{u}_b)dx$

Arbitrary virtual displacements: $\mathbf{F}_b = \mathbf{K}_b\mathbf{u}_b = \int_0^L \mathbf{T}_{sb}^{\mathrm{T}}\,\mathbf{K}_s\mathbf{T}_{sb}dx\,\mathbf{u}_b$

Displacement interpolation: $\mathbf{u}_s(x) = \mathbf{T}_{sb}(x)\mathbf{u}_b = \left[\left(\frac{4}{L}-\frac{6x}{L^2}\right) \quad \left(\frac{2}{L}-\frac{6x}{L^2}\right)\right]\begin{Bmatrix}u_{B1}\\ u_{B2}\end{Bmatrix}$

Result: $\mathbf{K}_b = \int_0^L \begin{Bmatrix}\left(\frac{4}{L}-\frac{6x}{L^2}\right)\\ \left(\frac{2}{L}-\frac{6x}{L^2}\right)\end{Bmatrix} EI\left[\left(\frac{4}{L}-\frac{6x}{L^2}\right) \quad \left(\frac{2}{L}-\frac{6x}{L^2}\right)\right]dx = \begin{bmatrix}\frac{4EI}{L} & \frac{2EI}{L}\\ \frac{2EI}{L} & \frac{4EI}{L}\end{bmatrix}$

Force Interpolation



Flexibility Matrix

Principle of virtual forces: $\delta \mathbf{F}_b^{\mathrm{T}} \mathbf{u}_b = \int_0^L \delta \mathbf{F}_s^{\mathrm{T}} \, \mathbf{u}_s dx$

Material law (flexibility, not stiffness): $\delta \mathbf{F}_b^{\mathrm{T}} \mathbf{u}_b = \int_0^L \delta \mathbf{F}_s^{\mathrm{T}} \, (\mathbf{f}_s \mathbf{F}_s) dx$

Equilibrium: $\delta \mathbf{F}_b^{\mathrm{T}} \mathbf{u}_b = \int_0^L \left(\delta \mathbf{F}_b^{\mathrm{T}} \widetilde{\mathbf{T}}_{sb}^{\mathrm{T}} \right) \mathbf{f}_s \left(\widetilde{\mathbf{T}}_{sb} \mathbf{F}_b \right) dx$

Arbitrary virtual force: $\mathbf{u}_b = \mathbf{f}_b \mathbf{F}_b = \int_0^L \widetilde{\mathbf{T}}_{sb}^{T} \, \mathbf{f}_s \widetilde{\mathbf{T}}_{sb} dx \; \mathbf{F}_b$

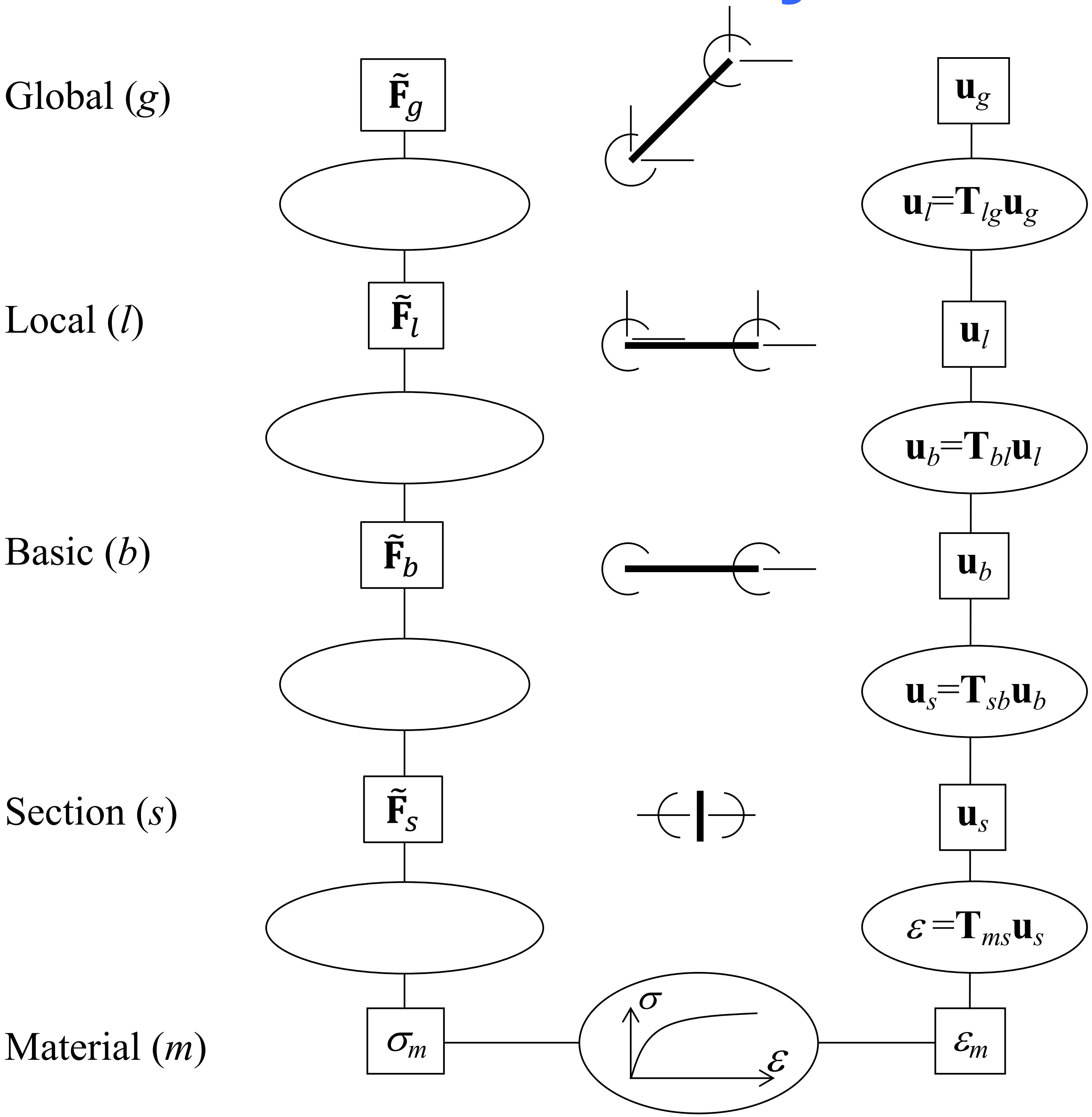
Force interpolation: $\mathbf{F}_s(x) = \widetilde{\mathbf{T}}_{sb}(x) \mathbf{F}_b = \left[-\left(1 - \frac{x}{L}\right) \quad \left(\frac{x}{L}\right) \right] \begin{Bmatrix} F_{b1} \\ F_{b2} \end{Bmatrix}$

Result: $\mathbf{f}_b = \int_0^L \begin{Bmatrix} -\left(1 - \frac{x}{L}\right) \\ \left(\frac{x}{L}\right) \end{Bmatrix} \frac{1}{EI} \left[-\left(1 - \frac{x}{L}\right) \quad \left(\frac{x}{L}\right) \right] dx = \begin{bmatrix} \frac{L}{3EI} & -\frac{L}{6EI} \\ -\frac{L}{6EI} & \frac{L}{3EI} \end{bmatrix}$

Two Sides of the Same Coin

$$\mathbf{f}_b = \mathbf{K}_b^{-1} \qquad \Longleftrightarrow \qquad \begin{bmatrix} \frac{L}{3EI} & -\frac{L}{6EI} \\ -\frac{L}{6EI} & \frac{L}{3EI} \end{bmatrix} = \begin{bmatrix} \frac{4EI}{L} & \frac{2EI}{L} \\ \frac{2EI}{L} & \frac{4EI}{L} \end{bmatrix}^{-1}$$

Distributed Plasticity Elements



Element Integration

Principle of virtual displacements: $\delta \mathbf{u}_b^{\mathrm{T}} \mathbf{F}_b = \int_0^L \delta \mathbf{u}_s^{\mathrm{T}} \, \mathbf{F}_s \, dx$

Kinematic compatibility: $\delta \mathbf{u}_b^{T} \mathbf{F}_b = \int_0^L \left(\delta \mathbf{u}_{\mathrm{b}}^{\mathrm{T}} \mathbf{T}_{sb}^{\mathrm{T}} \right) \mathbf{F}_s \, dx$

Arbitrary virtual displacements: $\mathbf{F}_b = \int_0^L \mathbf{T}_{sb}^{T} \, \mathbf{F}_s \, dx$

Displacement Interpolation

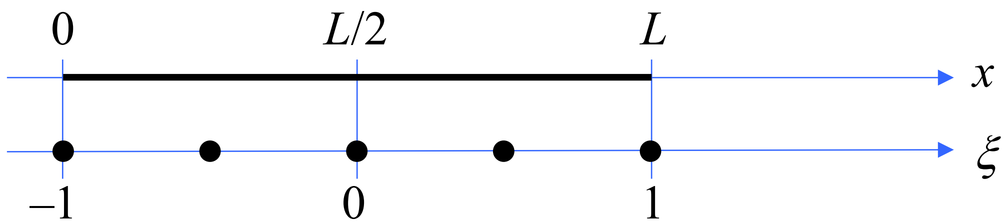
$$\mathbf{u}_s = \begin{Bmatrix} \varepsilon \\ \kappa \end{Bmatrix} = \begin{Bmatrix} u' \\ w'' \end{Bmatrix} = \begin{bmatrix} N_1{}' & 0 & 0 \\ 0 & N_2{}'' & N_3{}'' \end{bmatrix} \mathbf{u}_b = \mathbf{T}_{sb}\mathbf{u}_b$$

$$\begin{bmatrix} \dfrac{1}{L} & 0 & 0 \\ 0 & -\left(\dfrac{6 \cdot x}{L^2} - \dfrac{4}{L}\right) & \left(\dfrac{2}{L} - \dfrac{6 \cdot x}{L^2}\right) \end{bmatrix}$$

$$\widetilde{\boldsymbol{F}}_b = \int_0^L \mathbf{T}_{sb}^{\mathrm{T}}\, \widetilde{\boldsymbol{F}}_s\, dx + \overline{\boldsymbol{F}}_b$$

Quadrature

$$\widetilde{\boldsymbol{F}}_b = \int_0^L \mathbf{T}_{sb}^{\mathrm{T}}(x)\, \widetilde{\boldsymbol{F}}_s(x)\, dx = \int_{-1}^{1} \mathbf{T}_{sb}^{\mathrm{T}}(\xi)\, \widetilde{\boldsymbol{F}}_s(\xi) \left|\frac{dx}{d\xi}\right| d\xi = \sum_{i=1}^{N} \mathbf{T}_{sb}^{\mathrm{T}}(\xi_i)\, \widetilde{\boldsymbol{F}}_s(\xi_i) \cdot \frac{L}{2} \cdot weight_i$$



$$x = \frac{(\xi + 1) \cdot L}{2}$$

Rule	ξ-values (Lobatto integration)
2-point	–1, 1
3-point	–1, 0, 1
4-point	–1, –0.447, 0.447, 1
5-point	–1, –0.655, 0, 0.655, 1

Section Integration

Principle of virtual displacements: $\delta \mathbf{u}_s^{\mathrm{T}} \mathbf{F}_s = \int_A \delta\varepsilon \cdot \sigma \, dA$

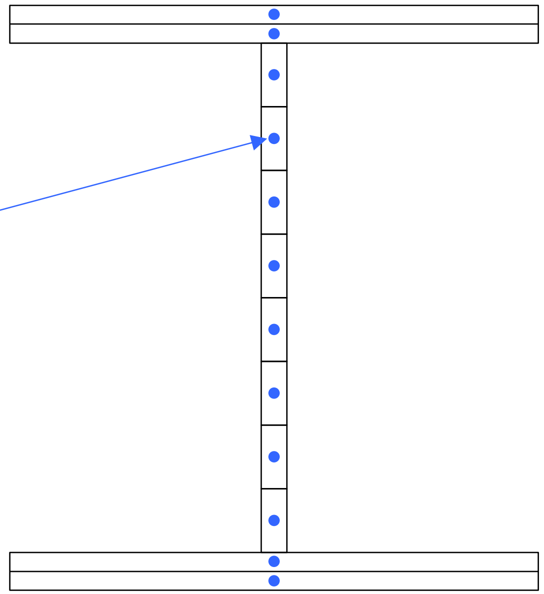
Kinematic compatibility: $\delta \mathbf{u}_s^{\mathrm{T}} \mathbf{F}_s = \int_A \left(\delta \mathbf{u}_s^{\mathrm{T}} \mathbf{T}_{ms}^{\mathrm{T}}\right) \cdot \sigma \, dA$

Arbitrary virtual displacements: $\mathbf{F}_s = \int_A \mathbf{T}_{ms}^{\mathrm{T}} \cdot \sigma \, dA$

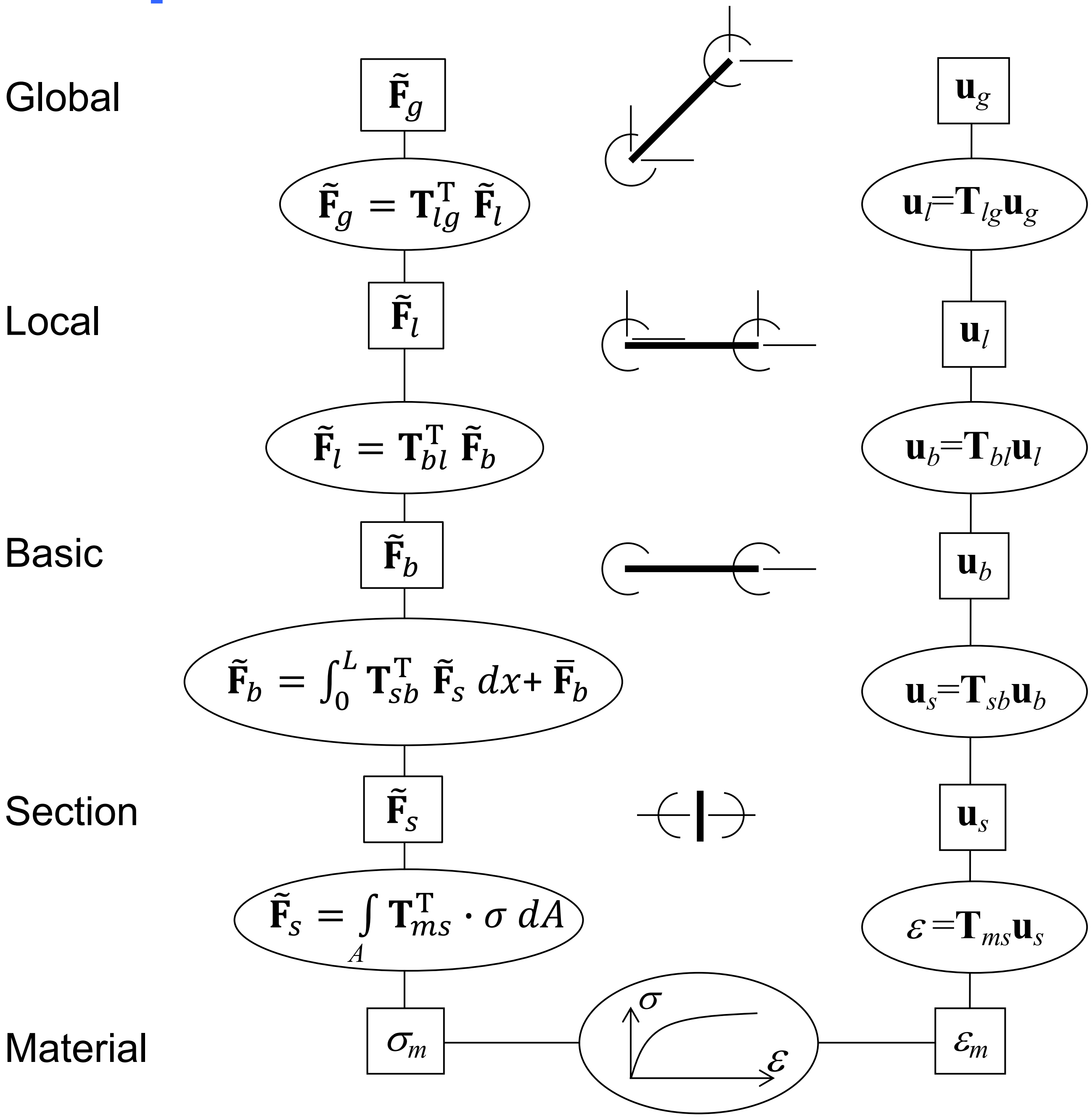
Fibres

$$\varepsilon_m = \mathbf{T}_{ms}\mathbf{u}_s = \{1, -z\}\begin{Bmatrix} \varepsilon \\ -\kappa \end{Bmatrix}$$

$$\widetilde{\boldsymbol{F}}_s = \int_A \mathbf{T}_{ms}^{\mathrm{T}} \cdot \sigma \, dA = \sum_{i=1}^{N} \begin{Bmatrix} 1 \\ -z \end{Bmatrix} \cdot \sigma_i$$



Displacement-based Element



State Determination

1. Trial displacements, $\mathbf{u}_f$

2. Strain, $\varepsilon = \mathbf{T}_{ms}\mathbf{T}_{sb}\mathbf{T}_{bl}\mathbf{T}_{lg}\left(\mathbf{T}_{ga}\mathbf{T}_{af}\right)\mathbf{u}_f$

3. Stress for given strain or strain increment

4. Resisting forces: $\tilde{\mathbf{F}}_f\left(\mathbf{u}_f\right) = \mathbf{T}_{af}^T \sum \left(\mathbf{T}_{ga}^T \mathbf{T}_{lg}^T \mathbf{T}_{bl}^T \left(\int_0^L \mathbf{T}_{sb}^{\mathrm{T}} \left(\int \mathbf{T}_{ms}^{\mathrm{T}} \cdot \sigma \, dA \right) dx + \bar{\mathbf{F}}_b \right) \right)$

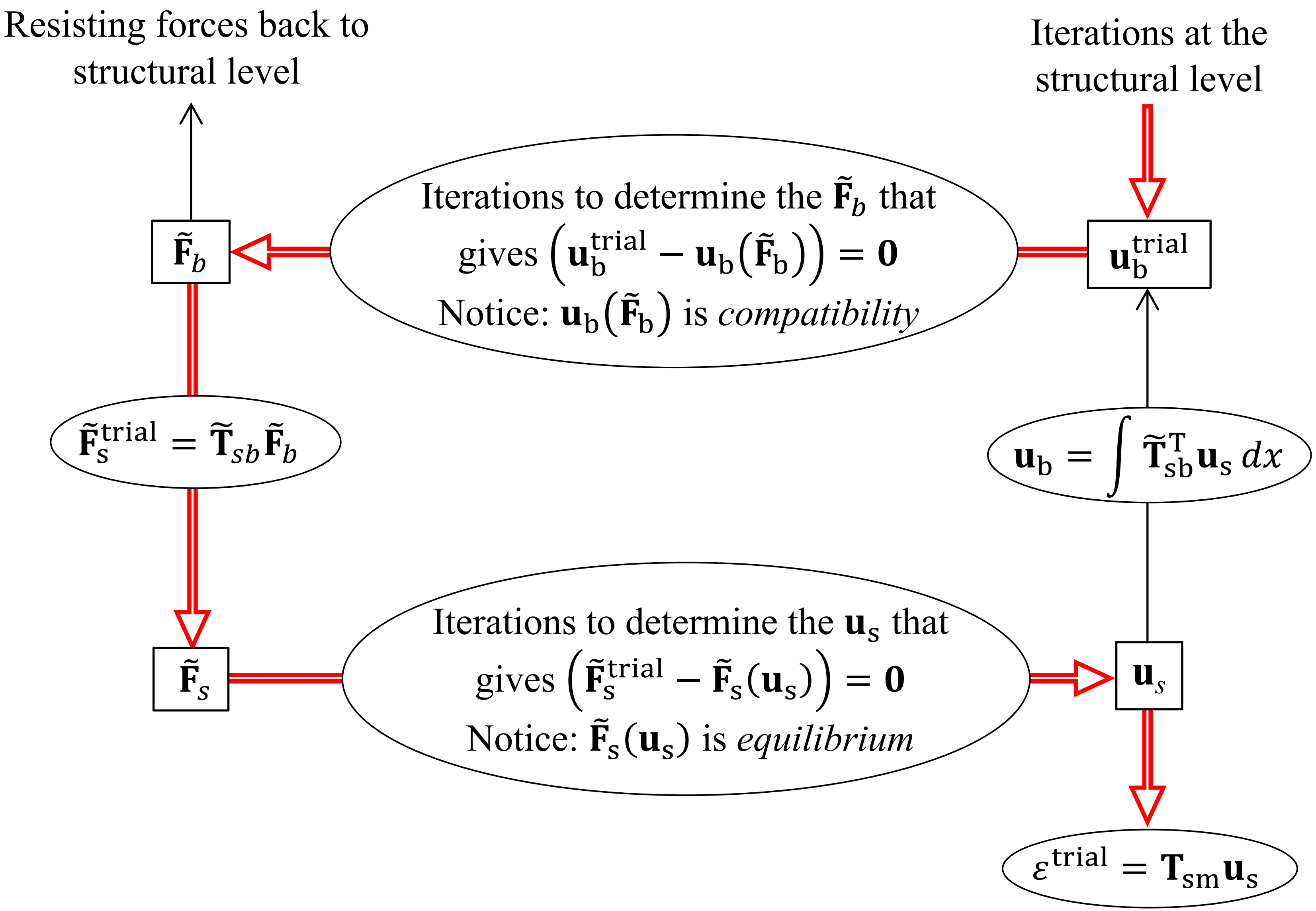
Force Interpolation

$$\widetilde{\boldsymbol{F}}_s = \widetilde{\boldsymbol{T}}_{sb}\widetilde{\boldsymbol{F}}_b$$

⇕

$$\begin{Bmatrix} N(x) \\ M(x) \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\left(1-\dfrac{x}{L}\right) & \dfrac{x}{L} \end{bmatrix} \begin{Bmatrix} N \\ M_1 \\ M_2 \end{Bmatrix}$$

State Determination Detour



Displacement-based vs. Force-based

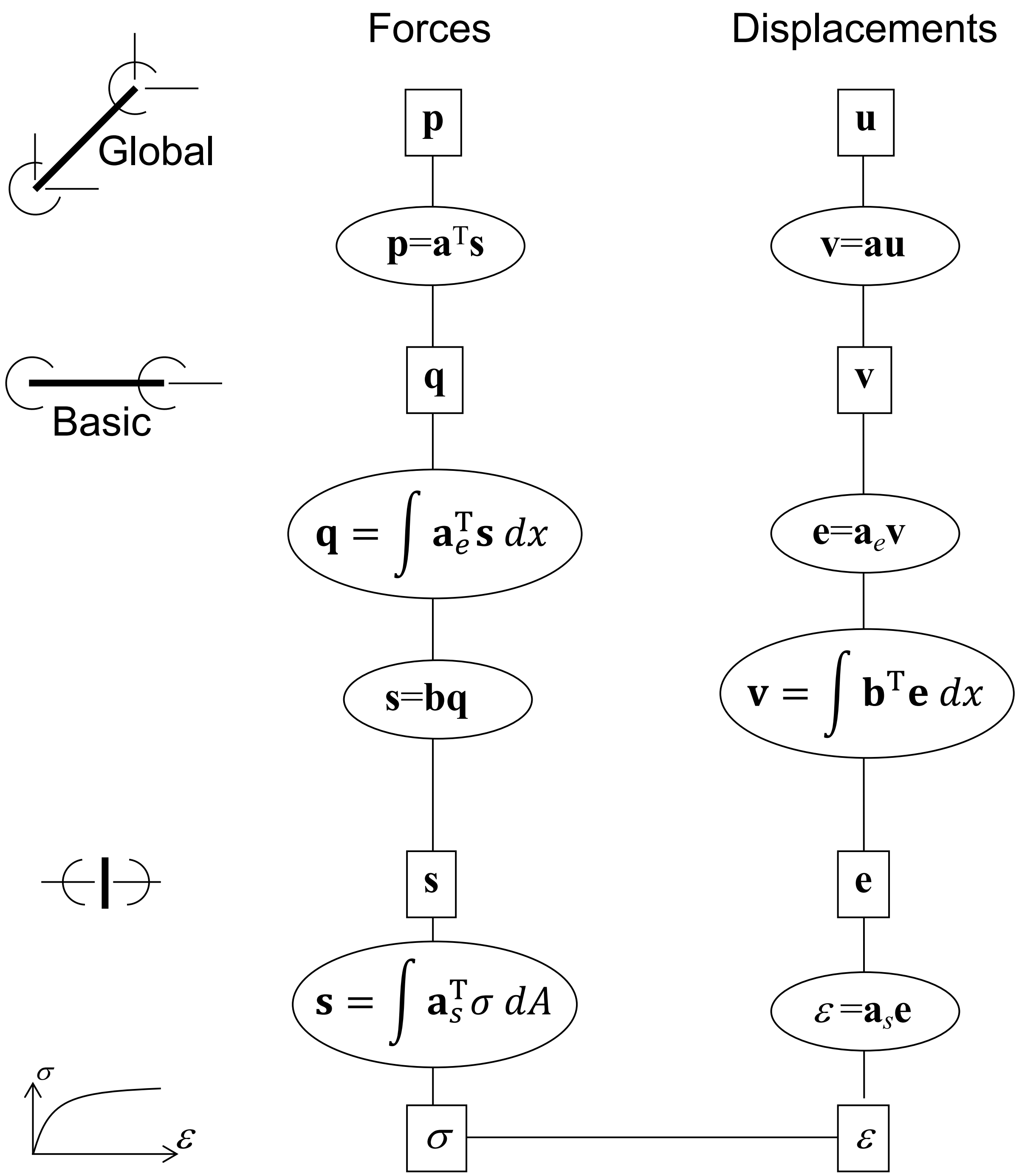
Displacement-based (G2 Element 12)

- Interpolates displacements (compatibility)
- Classical finite element approach
- Curvature is linear along element
- Require several elements along a member

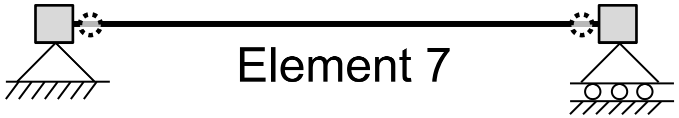
Force-based (G2 Element 13)

- Interpolates forces (equilibrium)
- Bending moment is linear along element
- Allows nonlinear variation in curvature
- One element may be sufficient along a member
- Requires iterations within the element

Berkeley Notation: OpenSees



Concentrated Plasticity Elements



Elastic perfectly plastic, i.e., end releases

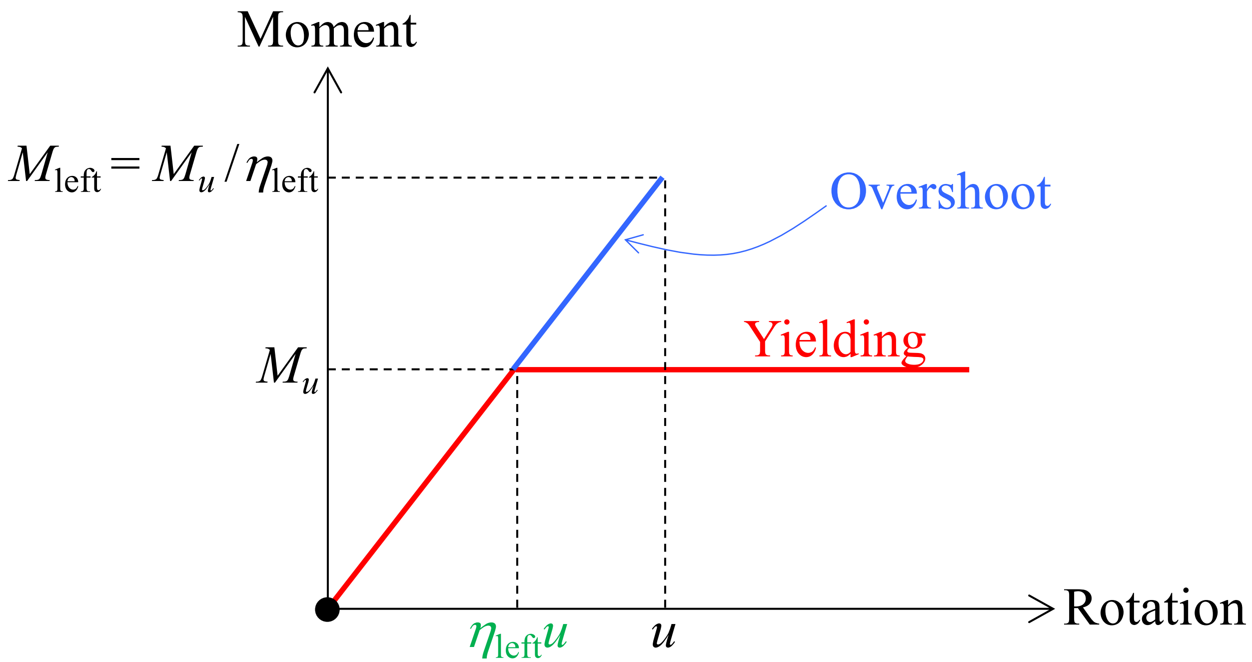
Two-component parallel system

One-component series system with uniaxial hysteric materials

Rigid interior with uniaxial hysteric material in springs

Finite hinge lengths

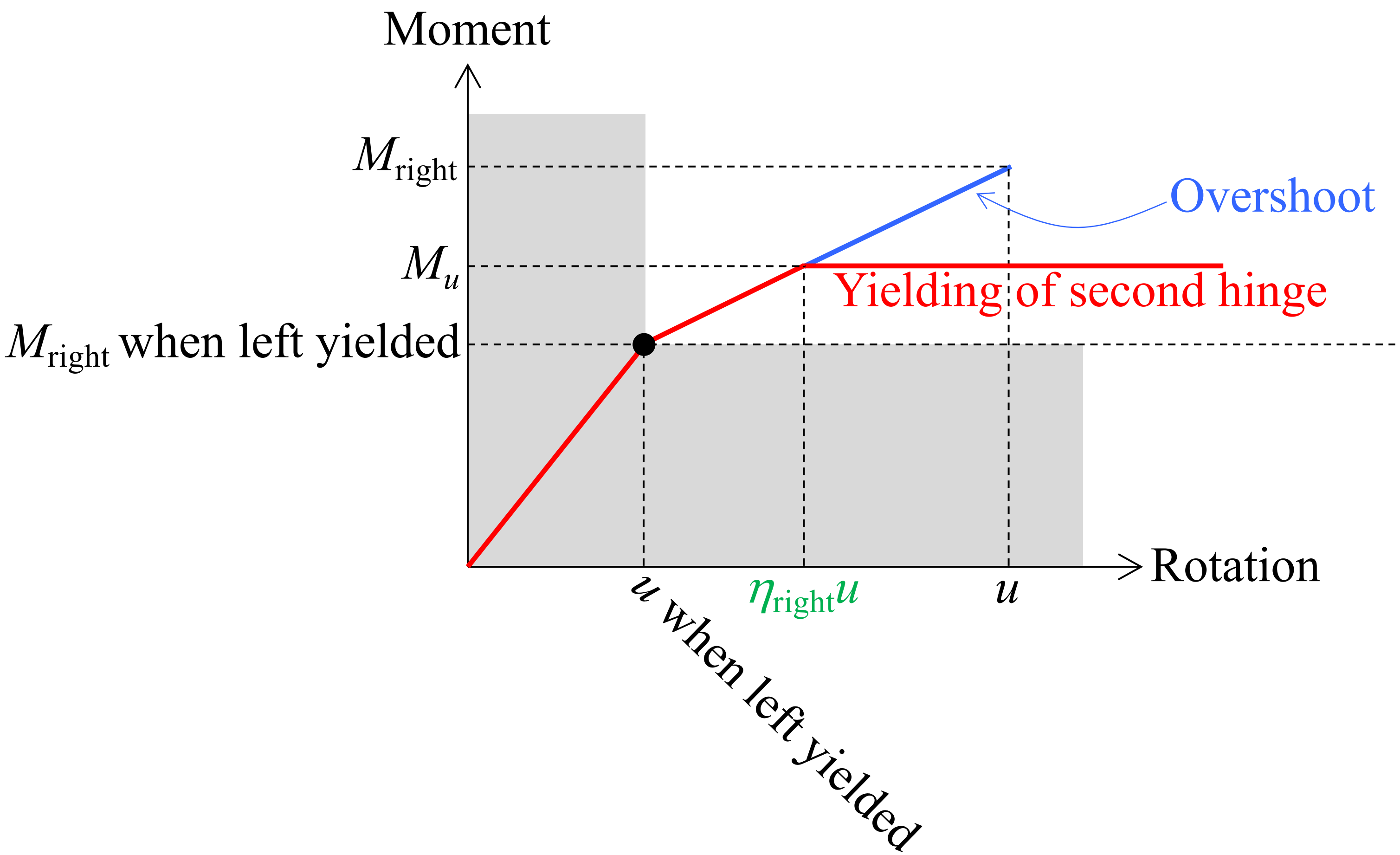
First Event Factor



$$\eta_{\mathrm{left}} = \frac{M_u}{M_{\mathrm{left}}}$$

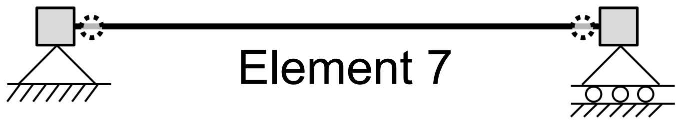
Second Event Factor

$$\eta_{\text{right}} = \frac{M_u - M_{\text{right when left yielded}}}{M_{\text{right}} - M_{\text{right when left yielded}}}$$



Element 7

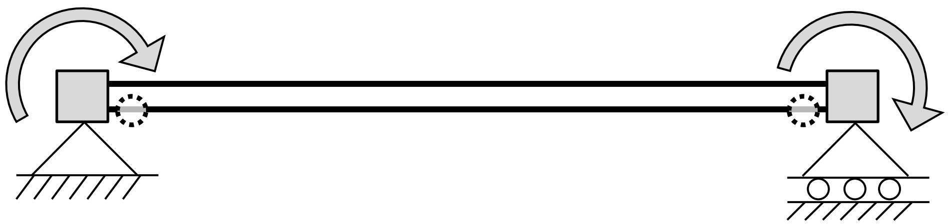
Elastic: $$\mathbf{F}_\mathrm{b} = \begin{bmatrix} \frac{4EI}{L} & \frac{2EI}{L} \\ \frac{2EI}{L} & \frac{4EI}{L} \end{bmatrix} \mathbf{u}_\mathrm{b}$$

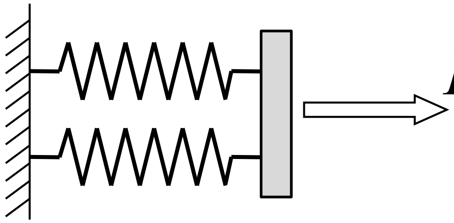


First yield: $$\mathbf{F}_\mathrm{b} = \eta_\mathrm{left} \cdot \begin{bmatrix} \frac{4EI}{L} & \frac{2EI}{L} \\ \frac{2EI}{L} & \frac{4EI}{L} \end{bmatrix} \mathbf{u}_\mathrm{b} + (1 - \eta_\mathrm{left}) \cdot \begin{bmatrix} 0 & 0 \\ 0 & \frac{3EI}{L} \end{bmatrix} \mathbf{u}_\mathrm{b}$$

Second yield: $$\mathbf{F}_\mathrm{b} = \eta_\mathrm{left} \cdot \begin{bmatrix} \frac{4EI}{L} & \frac{2EI}{L} \\ \frac{2EI}{L} & \frac{4EI}{L} \end{bmatrix} \mathbf{u}_\mathrm{b} + (1 - \eta_\mathrm{left}) \cdot \eta_\mathrm{right} \cdot \begin{bmatrix} 0 & 0 \\ 0 & \frac{3EI}{L} \end{bmatrix} \mathbf{u}_\mathrm{b} + (1 - \eta_\mathrm{left}) \cdot \left(1 - \eta_\mathrm{right}\right) \cdot \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \mathbf{u}_\mathrm{b}$$

Element 8





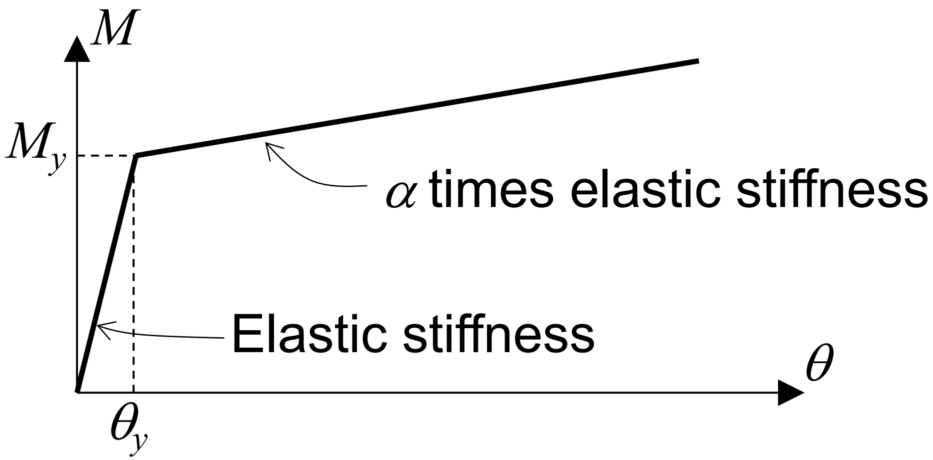
Parallel system:

$$u = u_1 = u_2$$

$$F = F_1 + F_2$$

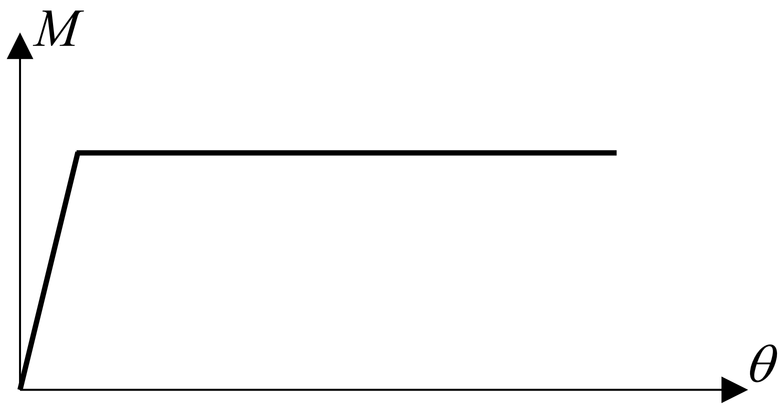
$$F = F_1 + F_2 = K_1 u_1 + K_2 u_2 = (K_1 + K_2)\, u$$

Sum of Stiffness



Elastic:

$$\mathbf{F}_b = \left(\alpha \begin{bmatrix} \frac{4EI}{L} & \frac{2EI}{L} \\ \frac{2EI}{L} & \frac{4EI}{L} \end{bmatrix} + (1-\alpha) \begin{bmatrix} \frac{4EI}{L} & \frac{2EI}{L} \\ \frac{2EI}{L} & \frac{4EI}{L} \end{bmatrix} \right) \mathbf{u}_b$$

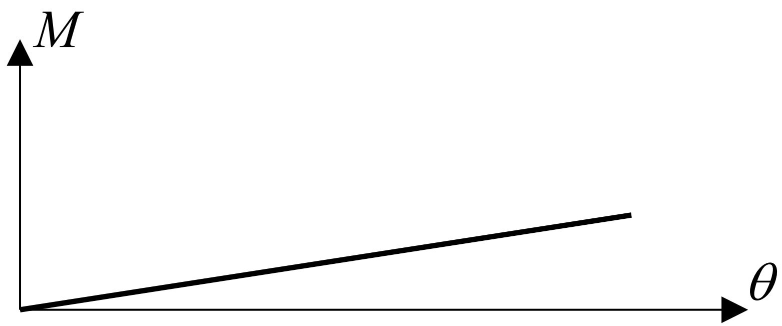


Yield left:

$$\mathbf{F}_b = \left(\alpha \begin{bmatrix} \frac{4EI}{L} & \frac{2EI}{L} \\ \frac{2EI}{L} & \frac{4EI}{L} \end{bmatrix} + (1-\alpha) \begin{bmatrix} 0 & 0 \\ 0 & \frac{3EI}{L} \end{bmatrix} \right) \mathbf{u}_b$$

Yield right:

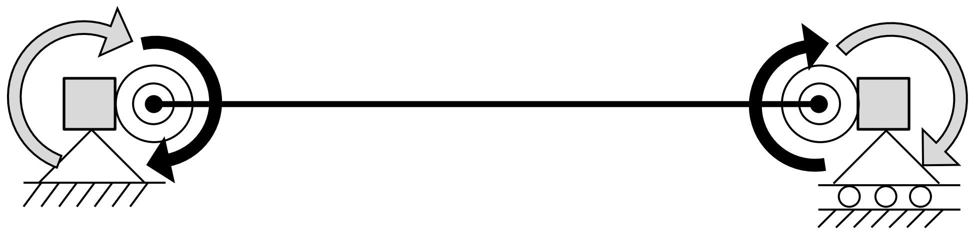
$$\mathbf{F}_b = \left(\alpha \begin{bmatrix} \frac{4EI}{L} & \frac{2EI}{L} \\ \frac{2EI}{L} & \frac{4EI}{L} \end{bmatrix} + (1-\alpha) \begin{bmatrix} \frac{3EI}{L} & 0 \\ 0 & 0 \end{bmatrix} \right) \mathbf{u}_b$$

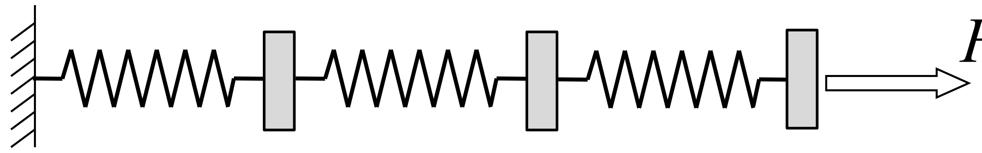


Yield both:

$$\mathbf{F}_b = \left(\alpha \begin{bmatrix} \frac{4EI}{L} & \frac{2EI}{L} \\ \frac{2EI}{L} & \frac{4EI}{L} \end{bmatrix} \right) \mathbf{u}_b$$

Element 9





Series system:

$$F = F_1 = F_2 = F_3$$

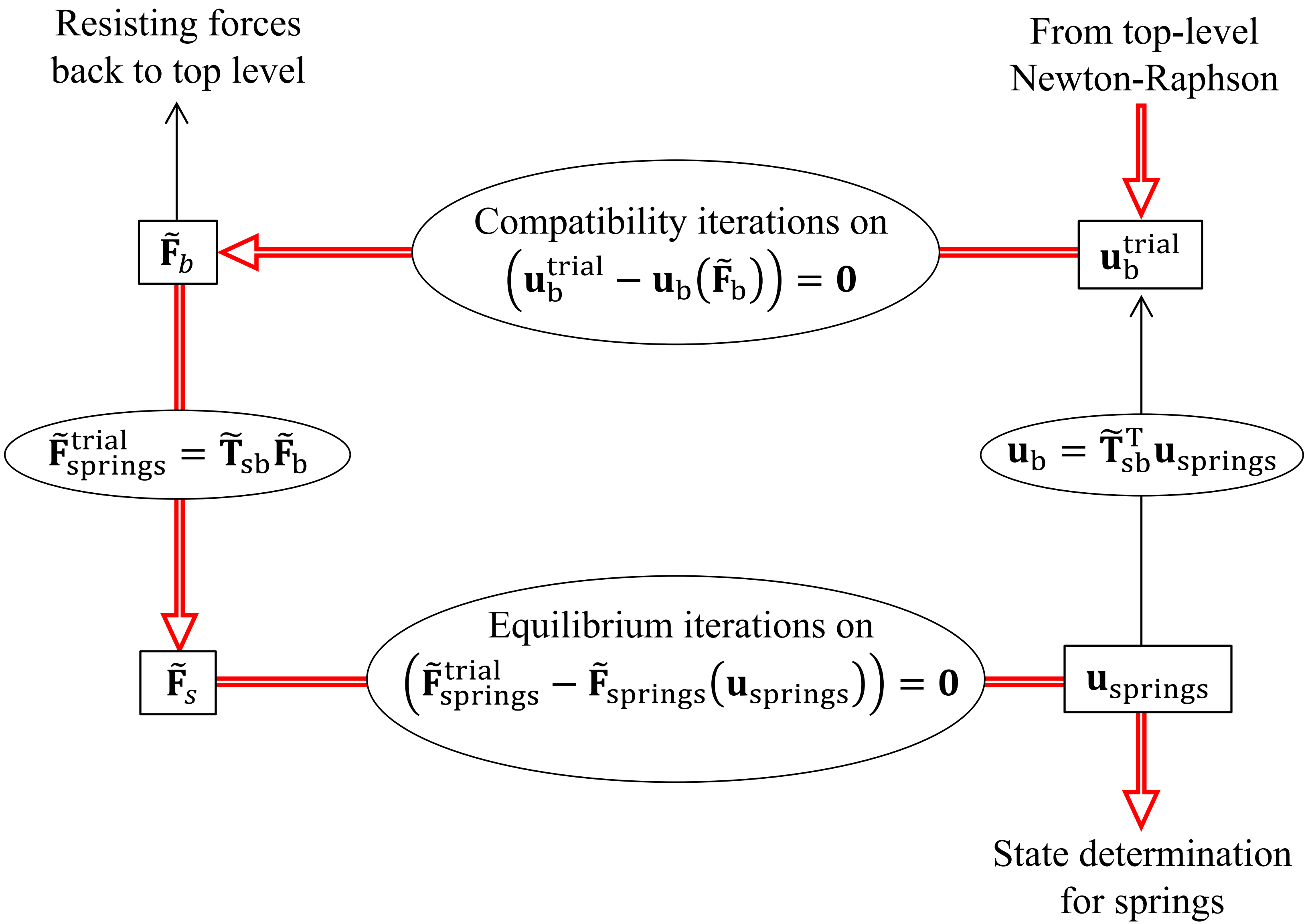
$$u{=}u_1{+}u_2{+}u_3$$

$$u = u_1 + u_2 + u_3 = f_1 F_1 + f_2 F_2 + f_3 F_3 = (f_1 + f_2 + f_3)\ F$$

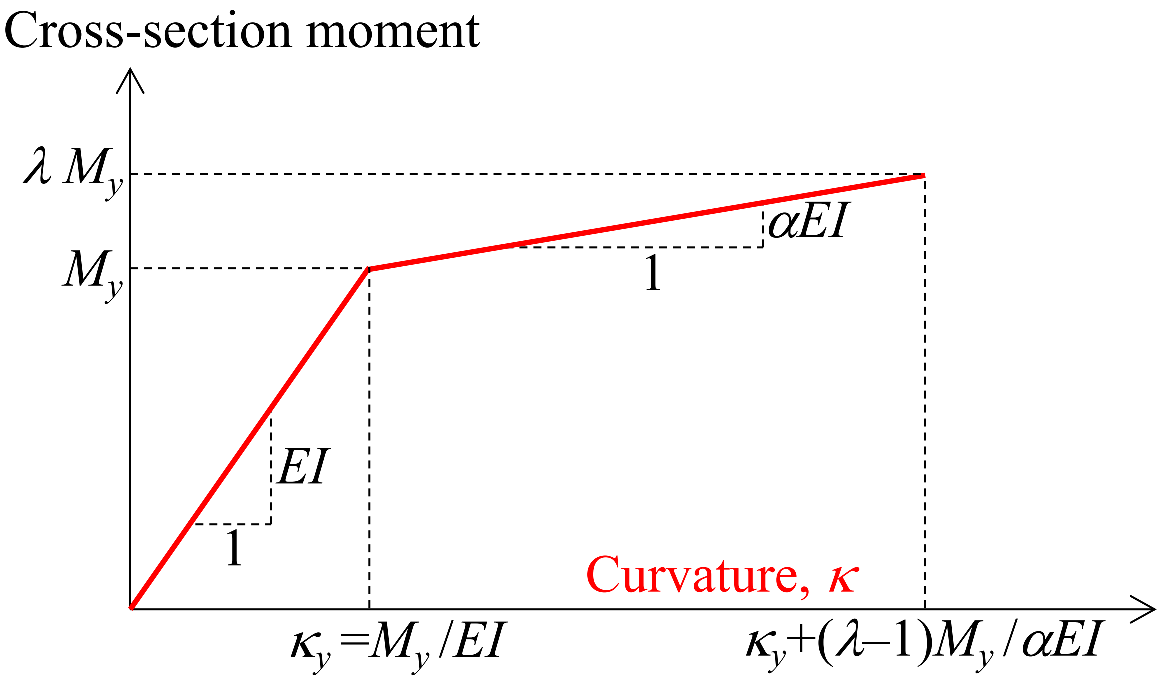
$$\mathbf{f}_b = \begin{bmatrix} \frac{L}{3EI} + f_{left} & -\frac{L}{6EI} \\ -\frac{L}{6EI} & \frac{L}{3EI} + f_{right} \end{bmatrix}$$

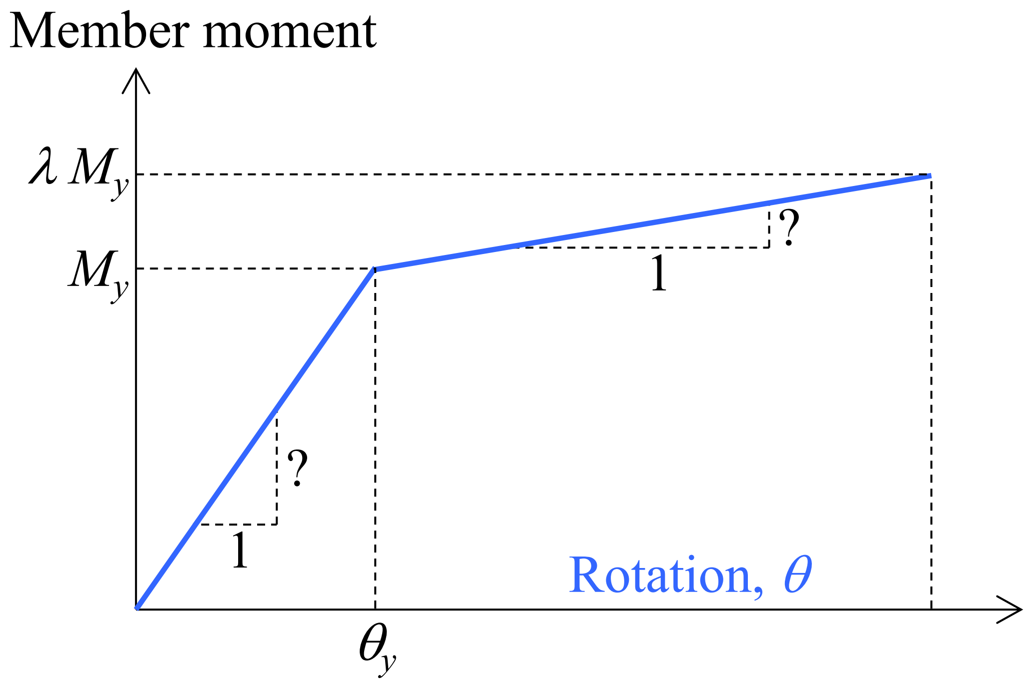
$$f_{\text{spring}} = \frac{L}{4 \cdot \alpha \cdot EI}$$

State Determination

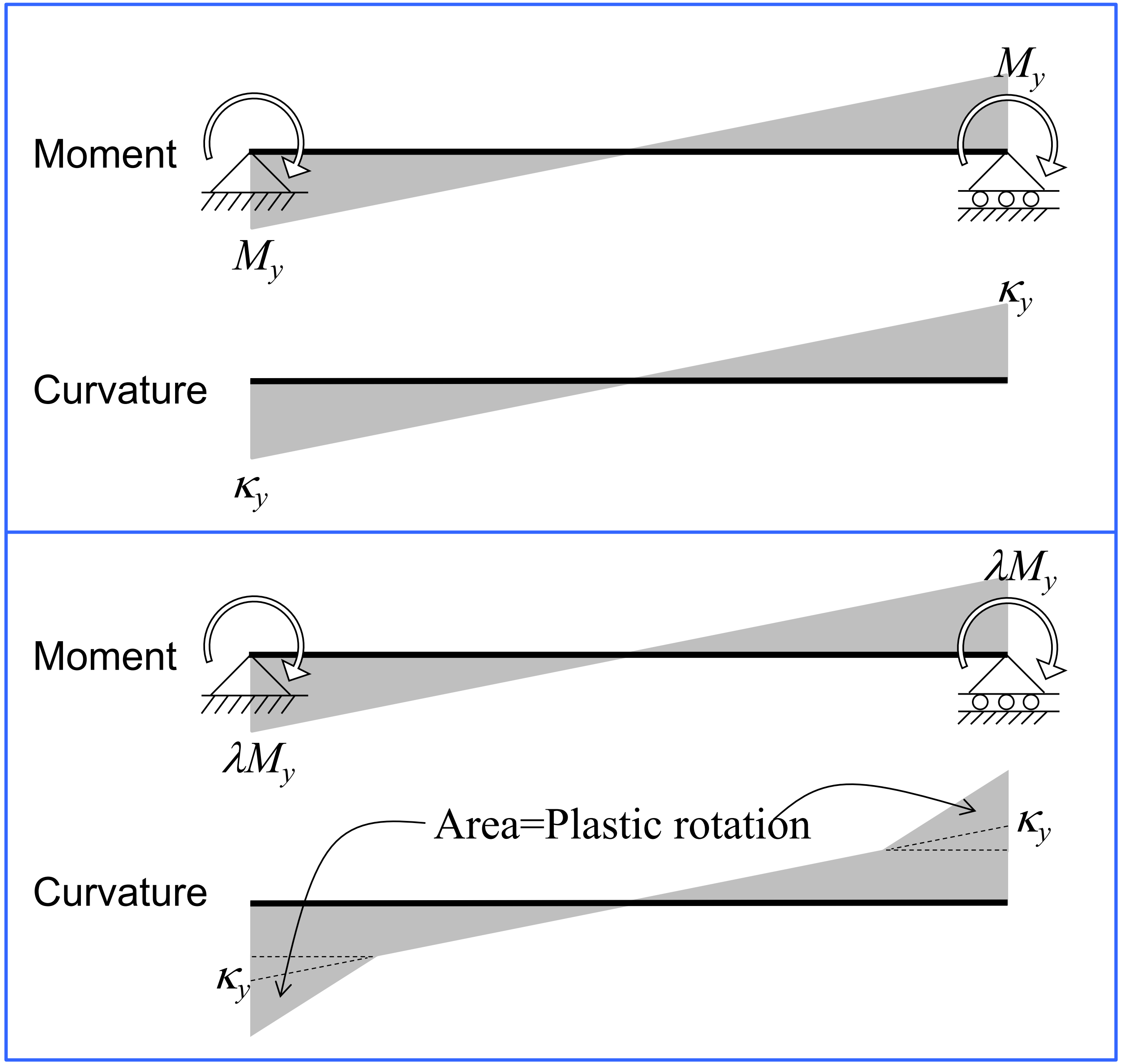


Spring Calibration





Plastic Rotation



More lectures:

Terje’s Toobox:

terje.civil.ubc.ca