A short course on

# Nonlinear Finite Element Analysis

This video:

**Geometric Nonlinearity**

Terje's Toolbox is freely available at terje.civil.ubc.ca
It is created and maintained by Professor Terje Haukaas, Ph.D., P.Eng.,
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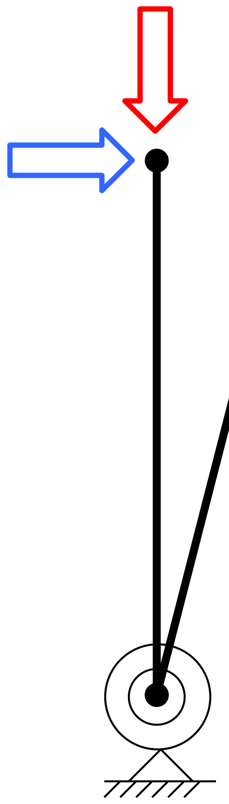
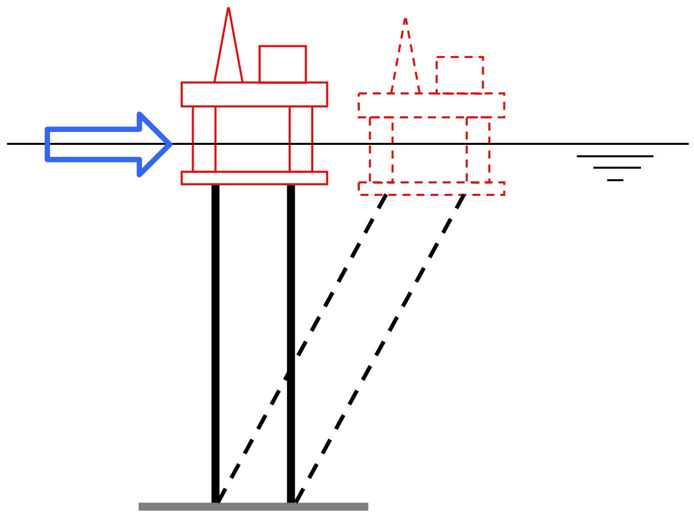
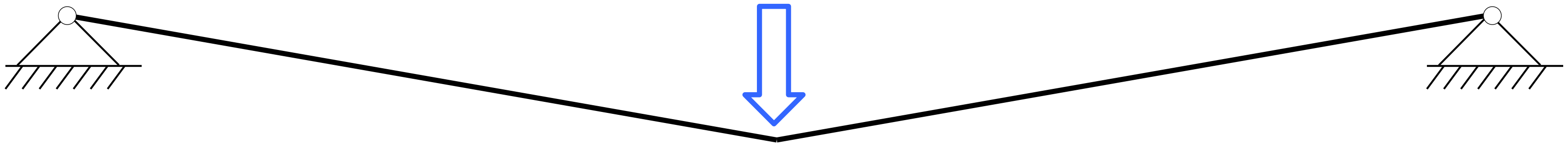
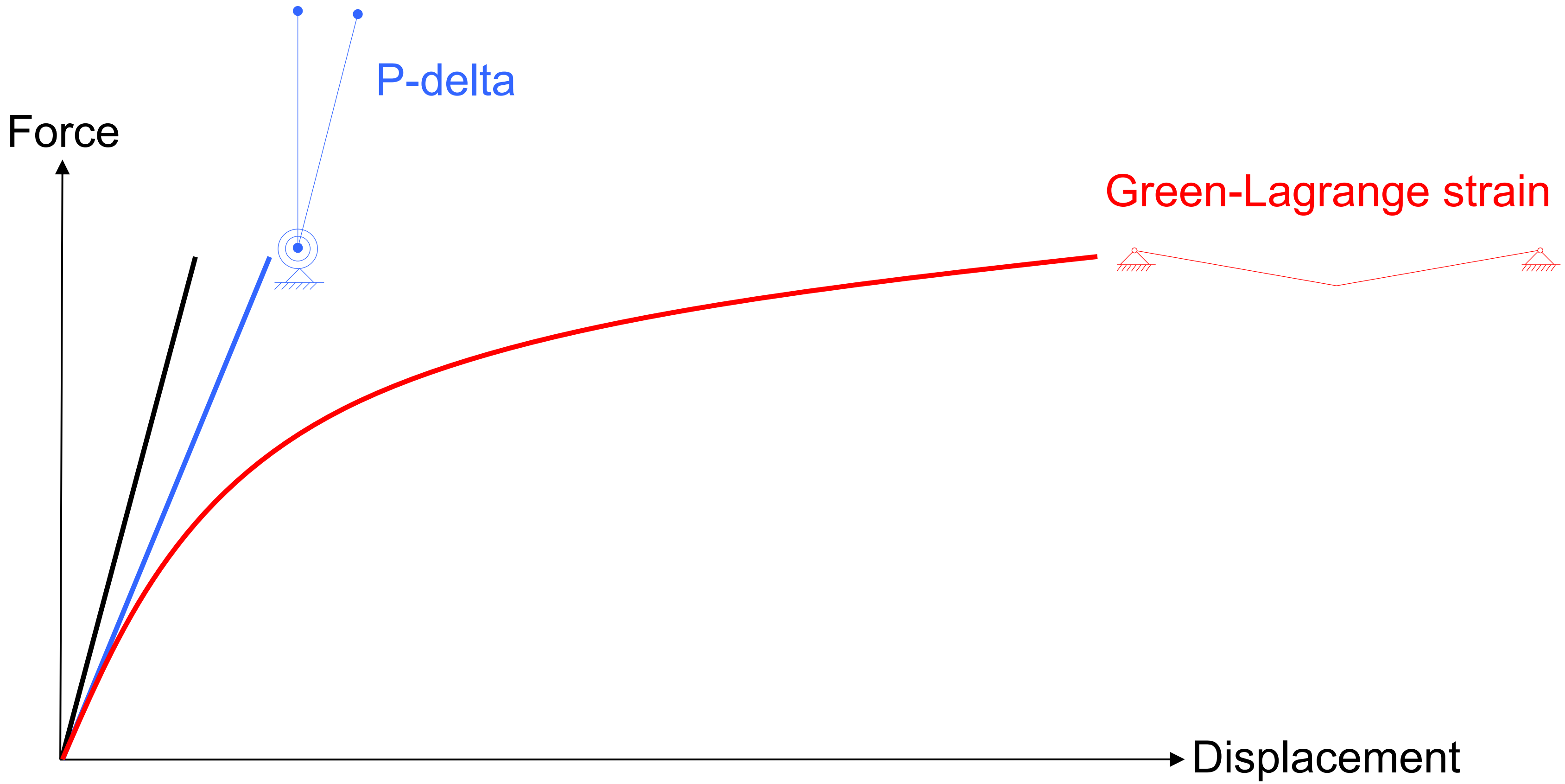
# Effects of Changing Geometry



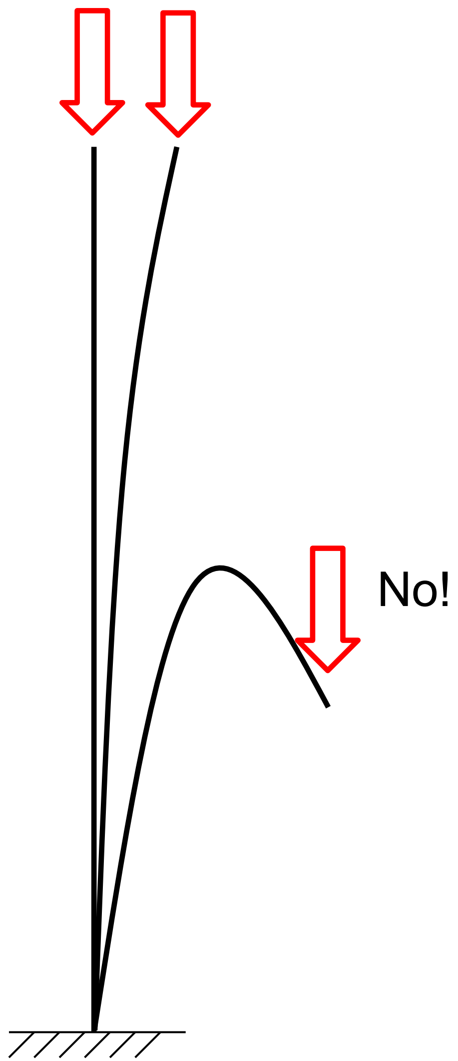




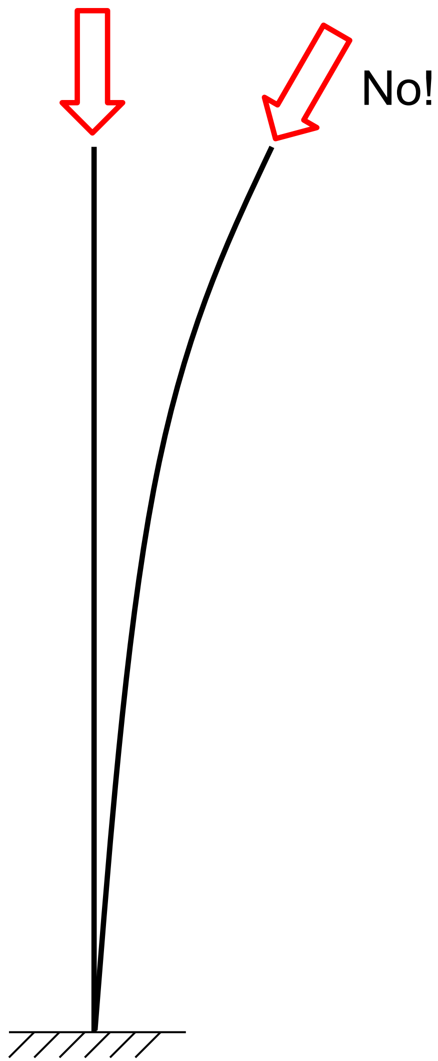
# Linear or Nonlinear



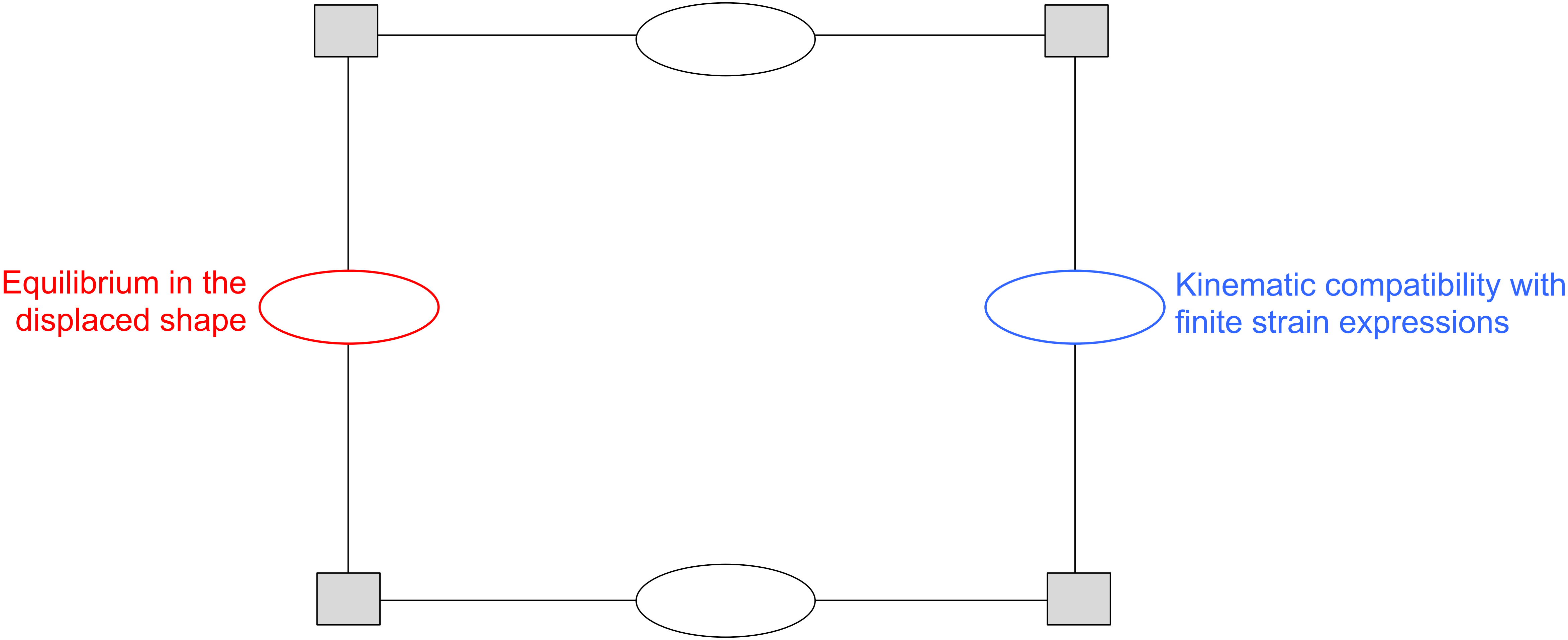
# Only Moderate Deformations



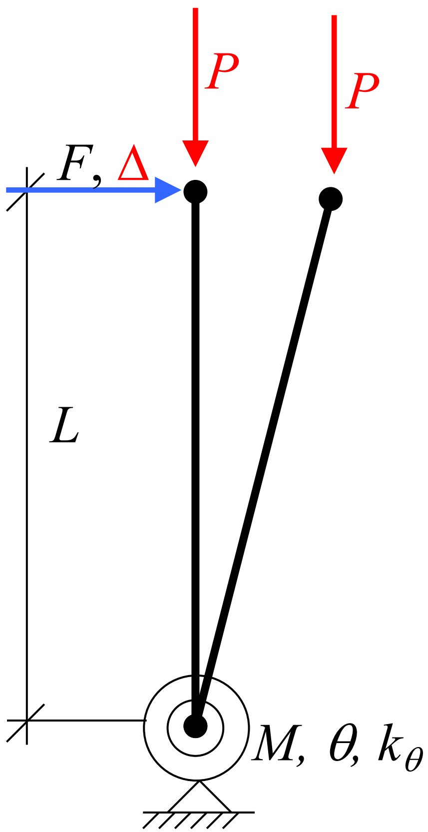
# Only Conservative Loads

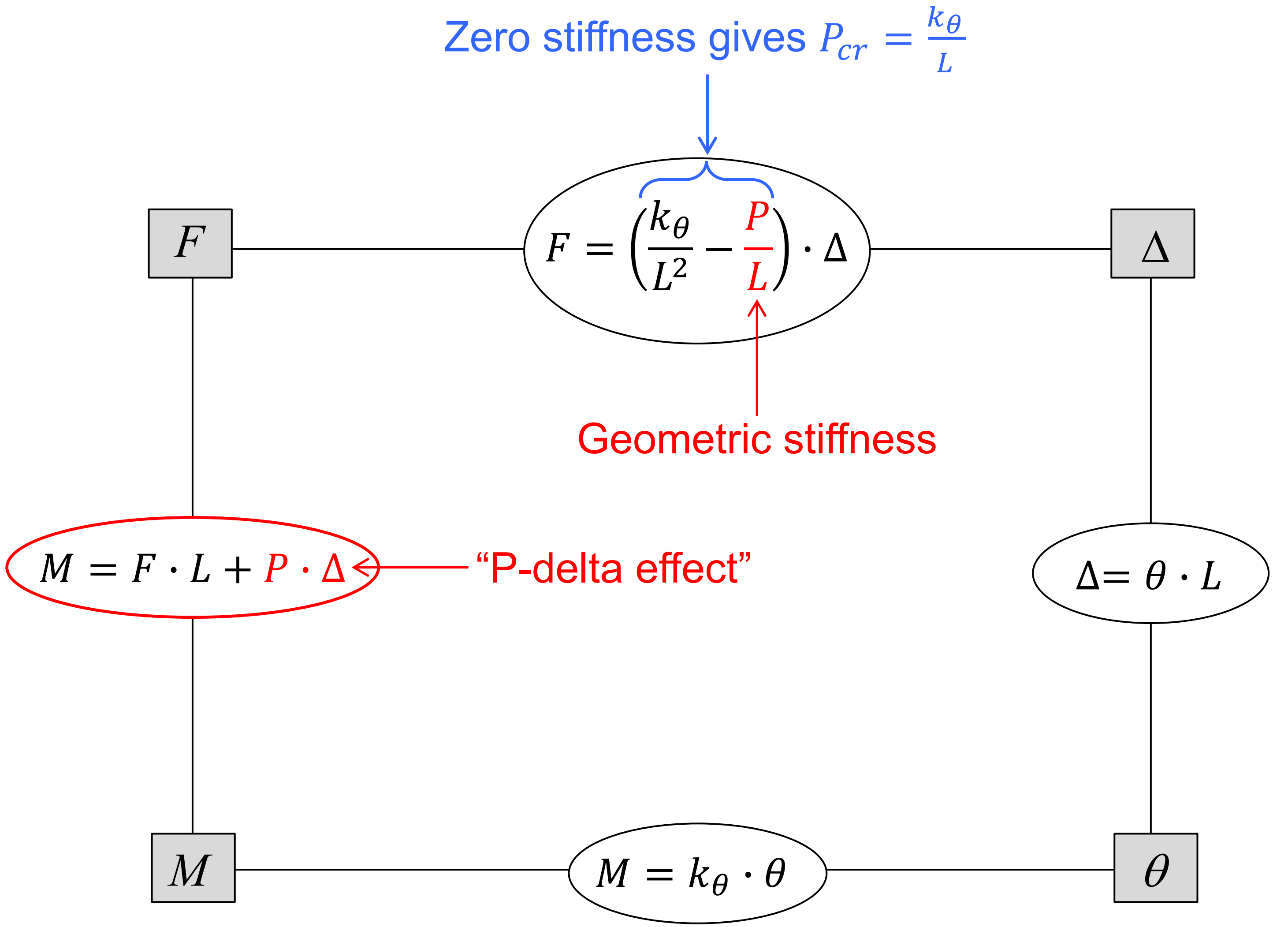


# How to Include



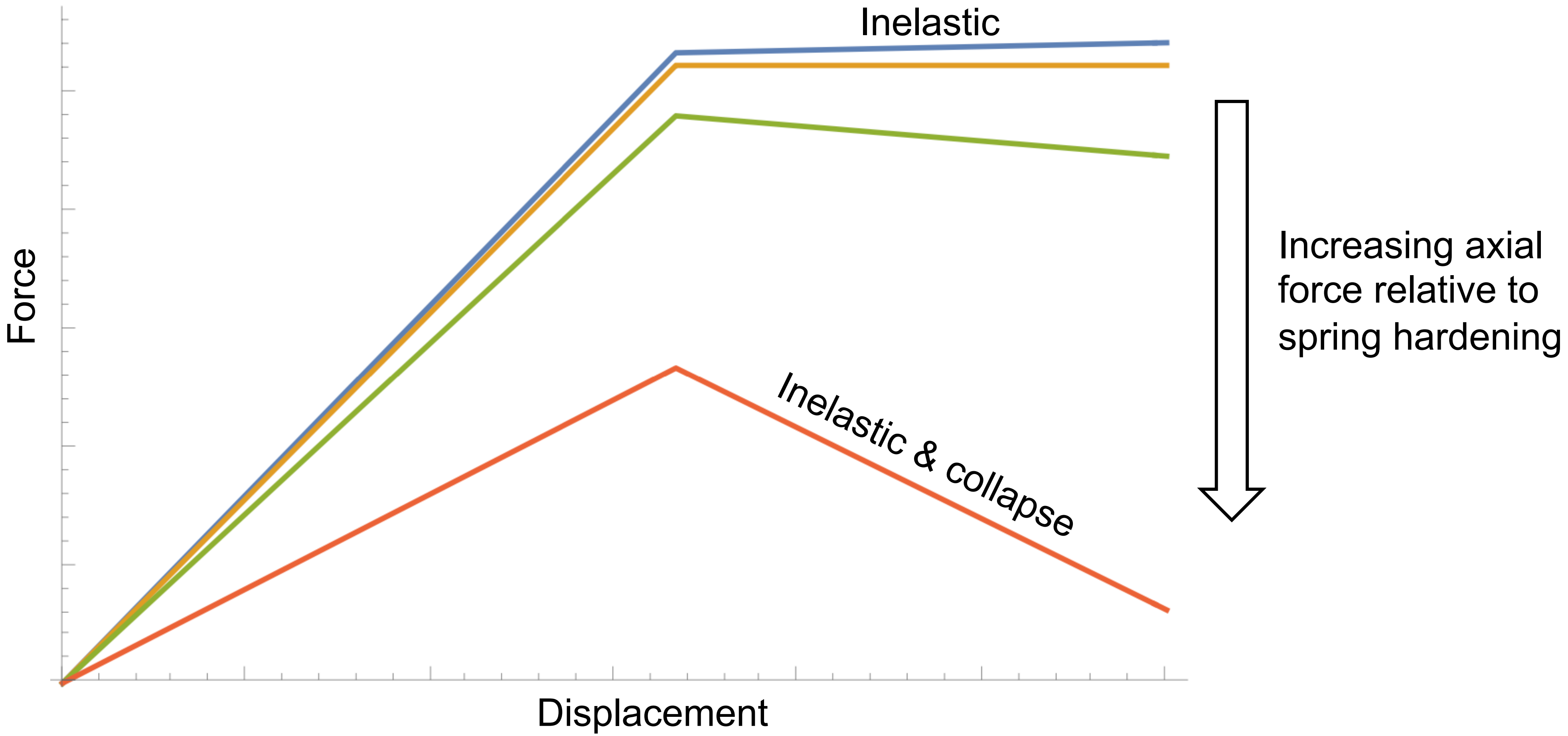
# Leaning Column



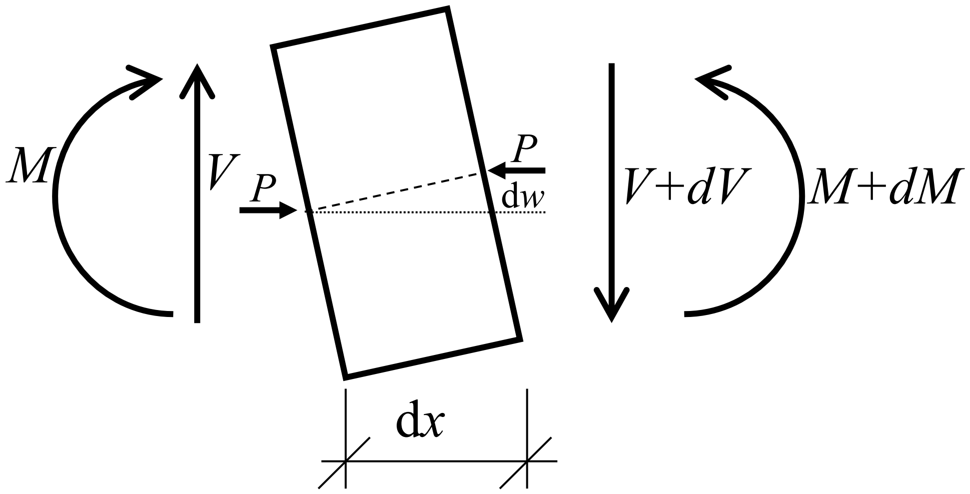



# Stick Model with P-Delta

See example at terje.civil.ubc.ca



# Beam with Axial Force




Equilibrium of displaced element:

$$V = \frac{dM}{dx} + P \cdot \frac{dw}{dx}$$

Revised differential equation:

$$EI \cdot \frac{d^4 w}{dx^4} + P \cdot \frac{d^2 w}{dx^2} = q_z$$

# Element 6

$$\int_0^L \left(EI \cdot w'''' + P \cdot w''\right)\delta w\, dx = 0$$

$$\int_0^L EI \cdot w''\delta w''dx - \int_0^L P \cdot w'\delta w' dx = 0$$

$$\left(\int_0^L EI \cdot \mathbf{N}''^T \cdot \mathbf{N}'' dx - P \cdot \underbrace{\int_0^L \mathbf{N}'^T \cdot \mathbf{N}' dx}_{\mathbf{k}_G}\right) \cdot \mathbf{u} = 0$$

$$\mathbf{K} = \begin{bmatrix} \frac{12EI}{L^3} & -\frac{6EI}{L^2} & -\frac{12EI}{L^3} & -\frac{6EI}{L^2} \\ -\frac{6EI}{L^2} & \frac{4EI}{L} & \frac{6EI}{L^2} & \frac{2EI}{L} \\ -\frac{12EI}{L^3} & \frac{6EI}{L^2} & \frac{12EI}{L^3} & \frac{6EI}{L^2} \\ -\frac{6EI}{L^2} & \frac{2EI}{L} & \frac{6EI}{L^2} & \frac{4EI}{L} \end{bmatrix} - P \cdot \underbrace{\begin{bmatrix} \frac{6}{5L} & -\frac{1}{10} & -\frac{6}{5L} & -\frac{1}{10} \\ -\frac{1}{10} & \frac{2L}{15} & \frac{1}{10} & -\frac{L}{30} \\ -\frac{6}{5L} & \frac{1}{10} & \frac{6}{5L} & \frac{1}{10} \\ -\frac{1}{10} & -\frac{L}{30} & \frac{1}{10} & \frac{2L}{15} \end{bmatrix}}_{\mathbf{K}_G}$$

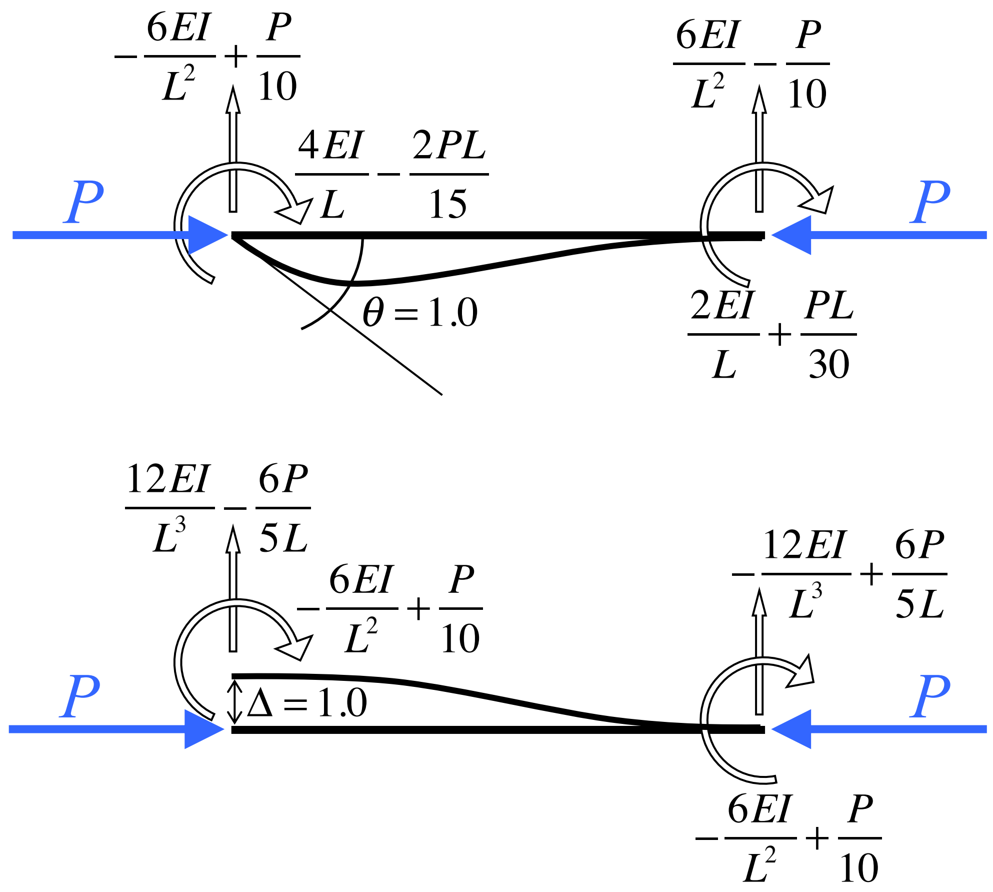

$$N_1(x) = \frac{2x^3}{L^3} - \frac{3x^2}{L^2} + 1$$

$$N_2(x) = -\frac{x^3}{L^2} + \frac{2x^2}{L} - x$$

$$N_3(x) = -\frac{2x^3}{L^3} + \frac{3x^2}{L^2}$$

$$N_4(x) = -\frac{x^3}{L^2} + \frac{x^2}{L}$$

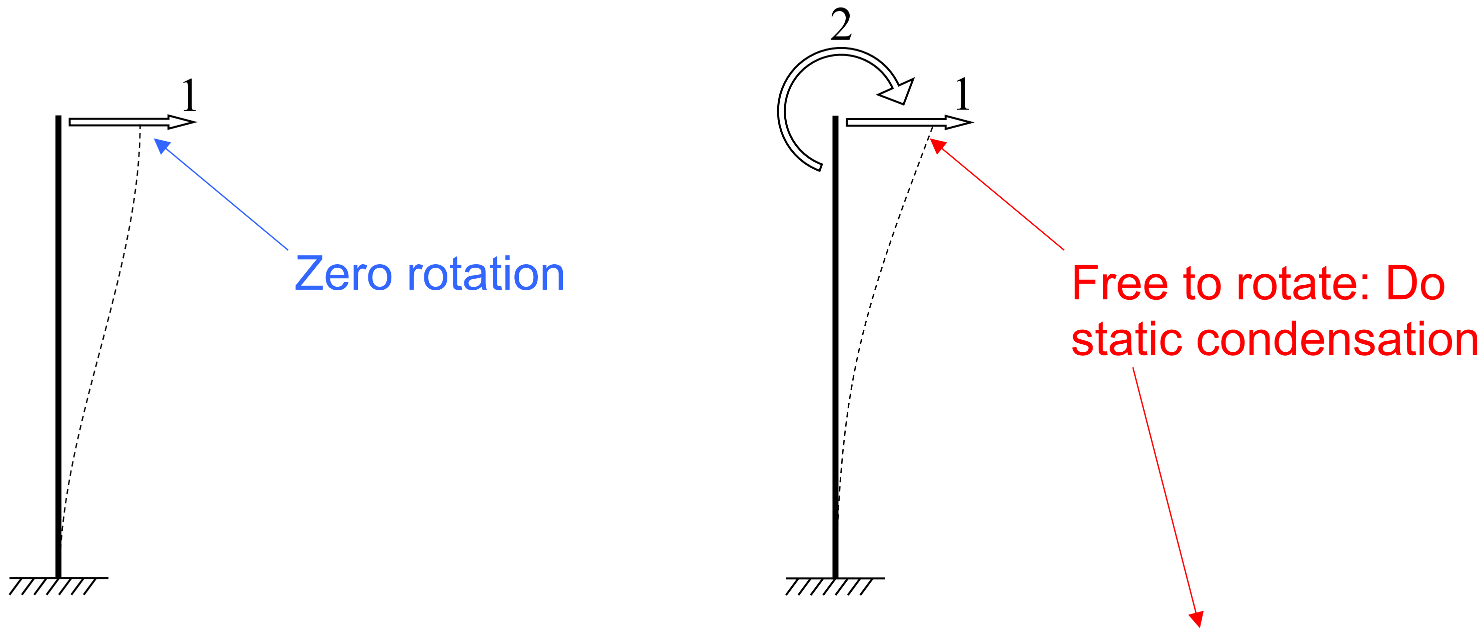
# Alternate View



# Big & Small P-Delta

$$\mathbf{K}_G = P\cdot\begin{bmatrix} \frac{1}{5L} & -\frac{1}{10} & -\frac{1}{5L} & -\frac{1}{10} \\ -\frac{1}{10} & \frac{2L}{15} & \frac{1}{10} & -\frac{L}{30} \\ -\frac{1}{5L} & \frac{1}{10} & \frac{1}{5L} & \frac{1}{10} \\ -\frac{1}{10} & -\frac{L}{30} & \frac{1}{10} & \frac{2L}{15} \end{bmatrix} + \begin{bmatrix} \frac{P}{L} & 0 & -\frac{P}{L} & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{P}{L} & 0 & \frac{P}{L} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

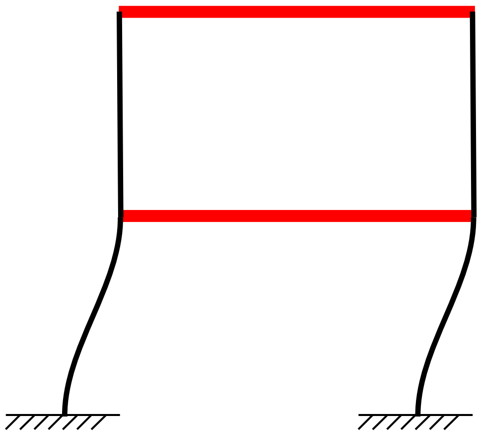
# Cantilever

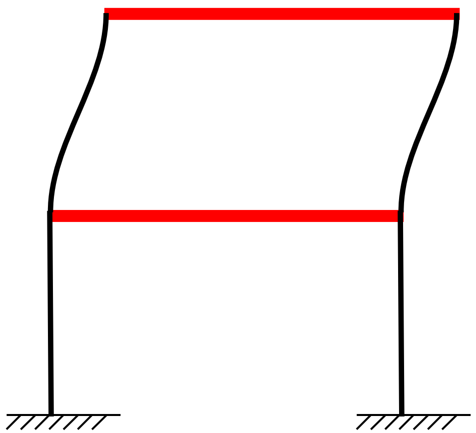



| | | |
|---|---|---|
| Elastic: | $K_o = \frac{12EI}{L^3}$ | $K_o = \frac{12EI}{L^3} - \left(-\frac{6EI}{L^2}\right)\left(\frac{L}{4EI}\right)\left(-\frac{6EI}{L^2}\right) = \frac{3EI}{L^3}$ |
| Big P-delta: | $K_G = -\frac{P}{L}$ | $K_G = -\frac{P}{L}$ |
| Small and big P-delta: | $K_G = -\frac{6P}{5L}$ | $K_G = \frac{6P}{5L} - \left(\frac{P}{10}\right)\left(-\frac{15}{2PL}\right)\left(\frac{P}{10}\right) = -\frac{9P}{8L}$ |

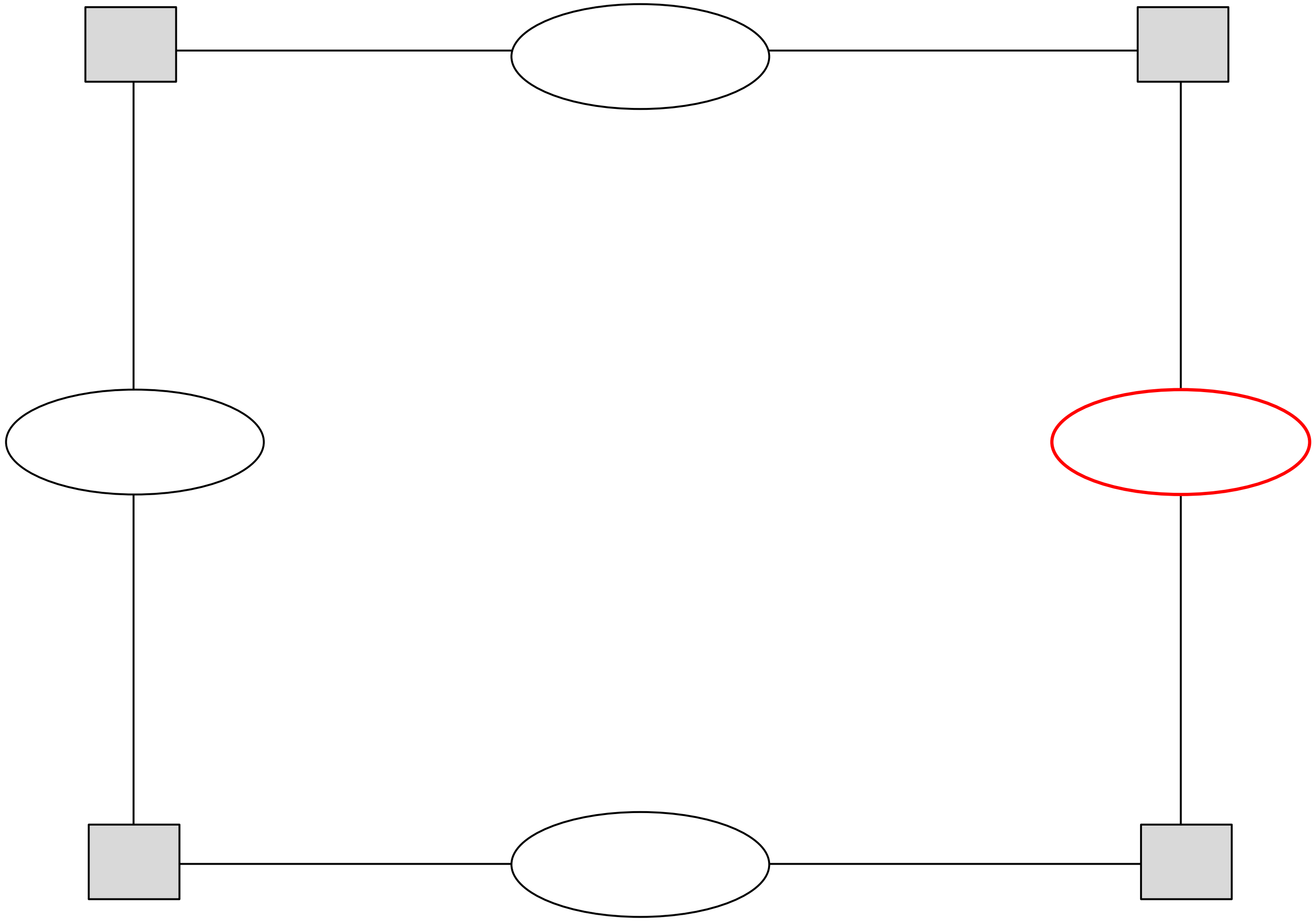
# Buckling Loads & Modes

Generalized eigenvalue problem: $[\mathbf{K} - P \cdot \mathbf{K}_G]\mathbf{u} = \mathbf{F}$





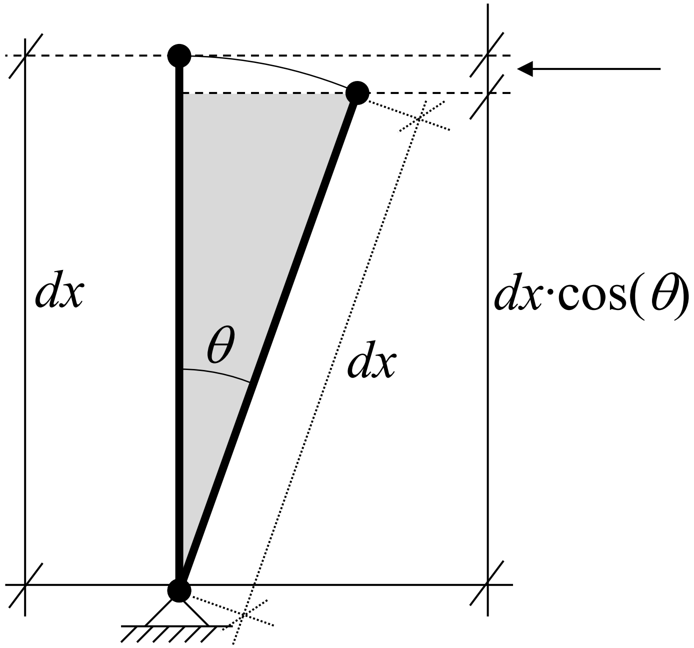
# Revised Compatibility



Finite strain expression
+ Principle of virtual displacements
= Finite elements with geometric nonlinearity

# Green-Lagrange Strain

$$E_{xx} = \frac{1}{2}\left(2 \cdot u_{x,x} + u_{x,x}u_{x,x} + u_{z,x}u_{z,x}\right) = \frac{du}{dx} + \frac{1}{2}\left(\frac{du}{dx}\right)^2 + \frac{1}{2}\left(\frac{dw}{dx}\right)^2$$




$$dx - dx\cdot\cos(\theta) = dx\cdot(1-\cos(\theta)) \approx \frac{1}{2}\cdot\theta^2$$

$$\cos(\theta) \approx 1 - \frac{\theta^2}{2}$$

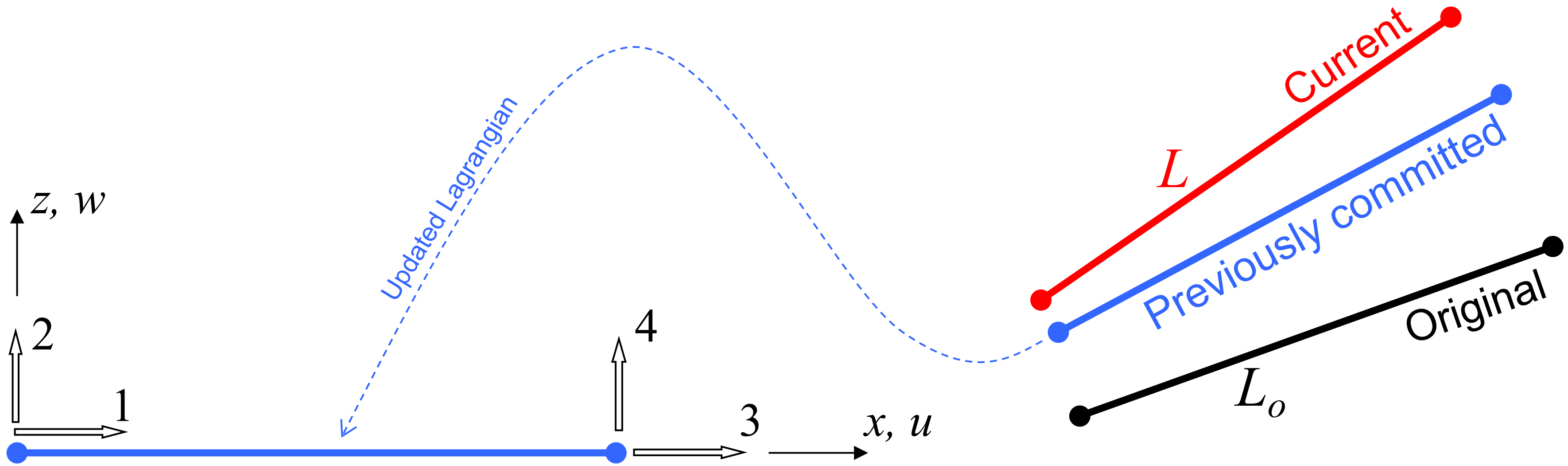
# Element 3

Green's strain:

$$\varepsilon = \frac{1}{2}\left(\left(\frac{L}{L_0}\right)^2 - 1\right) \leftrightarrow \varepsilon = \frac{du}{dx} + \frac{1}{2}\cdot\left(\frac{dw}{dx}\right)^2$$

Coordinate system:

Total Lagrangian **or** Updated Lagrangian **or** Corotational



# Basic to Local

Principle of virtual displacements: $\int_V \sigma\, \delta\varepsilon\, dV = \int_0^L \mathbf{p}^T\, \delta\tilde{\mathbf{u}}\, dx$

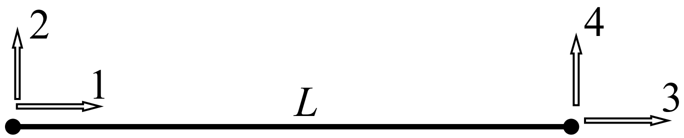
$N_1(x) = 1 - \frac{x}{L}$ and $N_2(x) = \frac{x}{L}$

Material law & section integration ($\sigma = E\varepsilon$): $EA \cdot \int_0^L \varepsilon\, \delta\varepsilon\, dx = \int_0^L \mathbf{p}^T\, \delta\tilde{\mathbf{u}}\, dx$

$$\varepsilon = u' + \frac{1}{2}\cdot w'^2$$

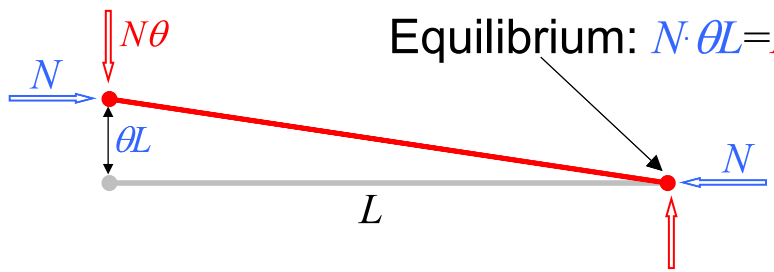
$$\begin{Bmatrix} u(x) \\ w(x) \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 \\ 0 & N_1 & 0 & N_2 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}$$

Variational calculus on Green's strain: $\delta\varepsilon = \delta\left(u' + \frac{1}{2}\cdot w'^2\right) = \delta u' + w' \cdot \delta w'$




Result: $\tilde{\mathbf{F}} = \begin{Bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{Bmatrix} N + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{L} & 0 & -\frac{1}{L} \\ 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{L} & 0 & \frac{1}{L} \end{bmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} N = \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} N + \begin{Bmatrix} 0 \\ -\left(\frac{u_4 - u_2}{L}\right) \\ 0 \\ \left(\frac{u_4 - u_2}{L}\right) \end{Bmatrix} N = \begin{Bmatrix} -1 \\ -\theta \\ 1 \\ \theta \end{Bmatrix} N$

Equilibrium: $N \cdot \theta L = FL$



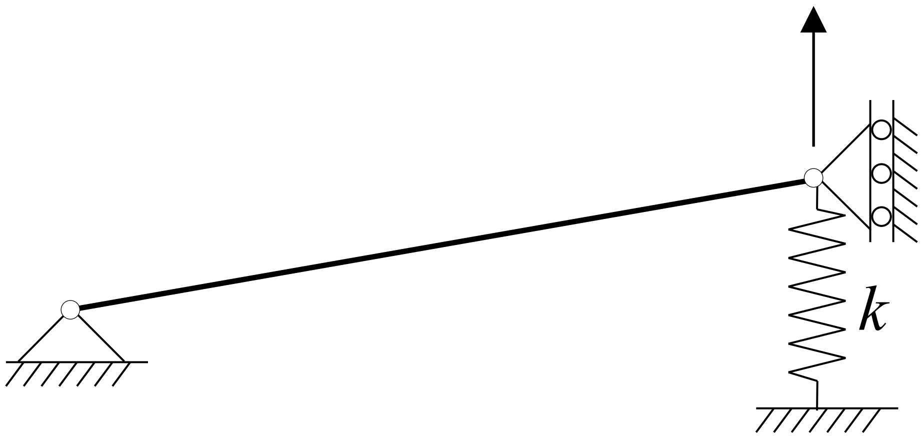

Equilibrium in the displaced shape and also $\mathbf{T}_{\mathrm{bl}}^{\mathrm{T}}$

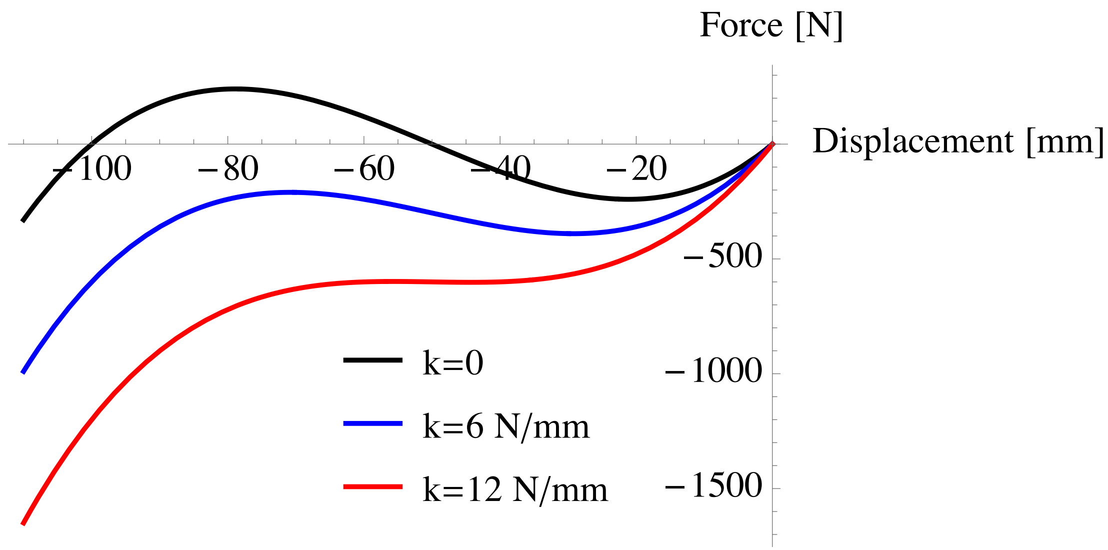
# Code

```python
# Original element length
dx0 = xyz[1,:] - xyz[0,:]
L0 = np.sqrt(dx0.dot(dx0))

# Current element length
du = U[[2, 3], 0] - U[[0, 1], 0]
dx = dx0 + du
L = np.sqrt(dx.dot(dx))

# Green-Lagrange strain
es = 0.5 * (L**2 / L0**2 - 1)

# Force and stiffness in the basic system
kb = self.E * self.A / L
self.sTrial = self.E * self.A * es + self.N0

# Previously committed geometry
duStep = U[[2, 3], 1] - U[[0, 1], 1]
dxCommitted = dx0 + du - duStep
LCommitted = np.sqrt(dxCommitted.dot(dxCommitted))

# Direction cosines for transformation matrices
dxCommitted = dxCommitted / LCommitted
duStep = duStep / L

# Rotation of member since last commit (cross-product of unit vectors is sin(theta)~theta for small rotations)
theta = (duStep).dot([-dxCommitted[1], dxCommitted[0]])

# Transformation from Basic to Local, with Local being last commit
Tbl = np.array([-1, -theta, 1, theta])

# Elastic element stiffness and force in Local configuration
Kl = np.outer(Tbl.dot(kb), Tbl)
tildeFl = Tbl.dot(self.sTrial)

# Add geometric stiffness matrix in the Local configuration
c = np.array([0, -1, 0, 1])
kGeometric = np.outer(c, c) * self.sTrial / L
Kl += kGeometric

# Transform from Local to Global
Tlg = np.array([[dxCommitted[0], dxCommitted[1], 0.0, 0.0],
              [-dxCommitted[1],  dxCommitted[0], 0.0, 0.0],
              [0.0, 0.0,         dxCommitted[0], dxCommitted[1]],
              [0.0, 0.0,        -dxCommitted[1], dxCommitted[0]]])
tildeFg = (np.transpose(Tlg)).dot(tildeFl)
Kg = ((np.transpose(Tlg)).dot(Kl)).dot(Tlg)
```

# Snap Through

See analytical example and also a G2 example at terje.civil.ubc.ca



# Forms of the BVP

$$EI \cdot w'''' - q = 0$$

Weight and integrate $\downarrow$ $\uparrow$ Require point-wise fulfilment

$$\int_0^L \left(EI \cdot w'''' - q\right) \cdot \delta w \cdot dx = 0$$

Integration by parts $\downarrow$ $\uparrow$ Integration by parts

$$\int_0^L EI \cdot w'' \, \delta w'' \, dx = \int_0^L q \cdot \delta w \, dx$$

Anti-variation $\downarrow$ $\uparrow$ Variation

$$\int_0^L \frac{1}{2} \cdot EI \cdot \left(w''\right)^2 dx = \int_0^L q \cdot w \, dx$$

# Energy Principles

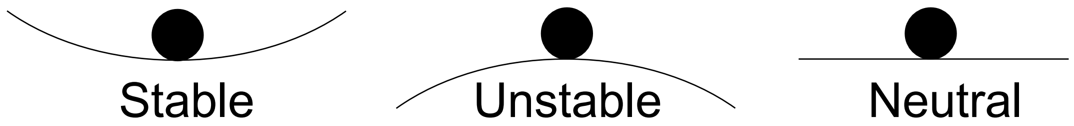
Total energy = Strain energy + Potential energy in applied loads

$$\Pi = U + H$$

Functionals: $\Pi(w(x)) = U(w(x)) - H(w(x))$

Principle of minimum potential energy: $\delta\Pi = 0$

Stability: $\delta^2\Pi > 0$



# Ritz

Guess $w(x)$ that satisfies the "essential" kinematic boundary conditions

$$w(x) = a \cdot \left(1 - \cos\left(\frac{n \cdot \pi \cdot x}{2\,L}\right)\right)$$

$$w(x) = a \cdot \left(\sin\left(\frac{n \cdot \pi \cdot x}{2\,L}\right)\right)$$

$$w(x) = \sum a_n \cdot \left(1 - \cos\left(\frac{n \cdot \pi \cdot x}{2\,L}\right)\right)$$

Principle of minimum potential energy + calculus of variation:

$$\delta\Pi = \frac{\partial\Pi}{\partial a} \cdot \delta a = 0$$

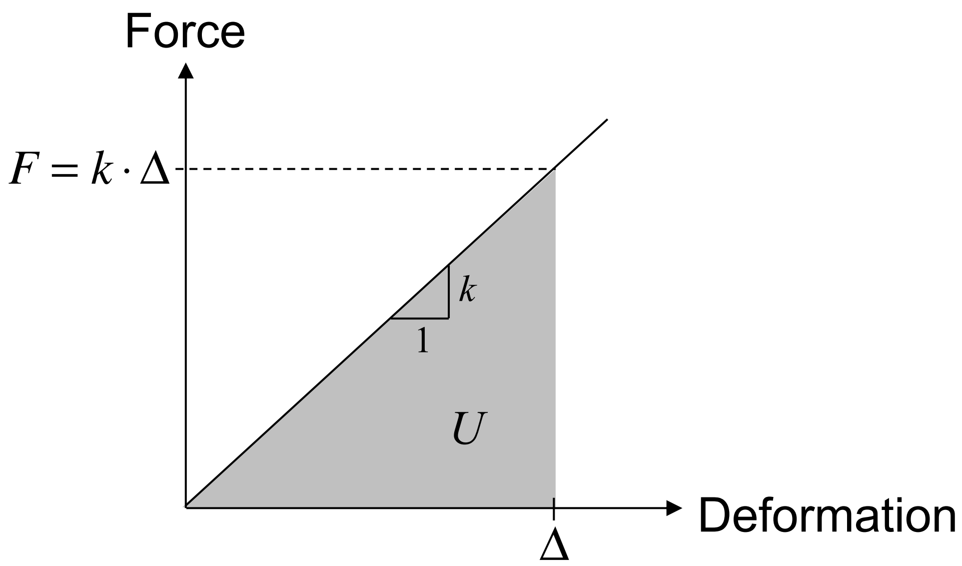
Must be zero!

Arbitrary variations!

Solve for $a$, or solve for the buckling load if $a$ cancels

# Energy Expressions

- In general: $U = \int_V \frac{1}{2} \cdot E \cdot (\varepsilon)^2 \, dV$ $\quad \varepsilon = z \cdot \frac{d^2 w}{dx^2} + \frac{1}{2} \cdot \left(\frac{dw}{dx}\right)^2$
- Elastic spring: $U = \int_0^\Delta F \, d\Delta = \int_0^\Delta (k \cdot \Delta) d\Delta = \frac{1}{2} \cdot k \cdot \Delta^2$
- Beam bending: $U = \int_0^{w''} EI \cdot w'' dw'' = \frac{1}{2} \cdot EI \cdot (w'')^2$
- Point load: $H = -F \, \Delta$
- Distributed load: $H = -\int_0^L q_z \cdot w \, dx$
- Axial force: $H = -\int_0^L P \, du = -P \cdot \int_0^L \frac{1}{2} \cdot (w')^2 \, dx$



# Buckling of Simply Supported Column

See example at terje.civil.ubc.ca

$$U = \int_0^{w''} EI \cdot w'' dw'' = \frac{1}{2} \cdot EI \cdot (w'')^2$$

$$H = -\int_0^L P\, du = -P \cdot \int_0^L \frac{1}{2} \cdot (w')^2\, dx$$

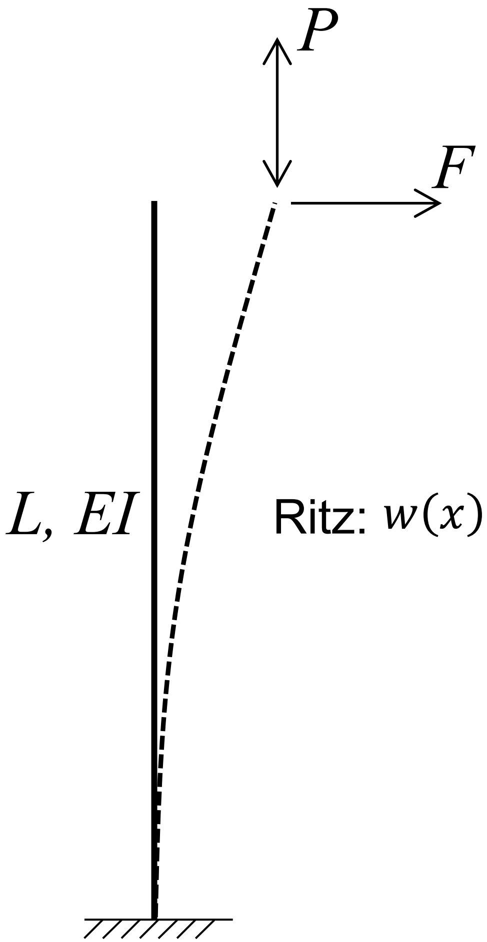
$$w(x) = a \cdot \sin\left(\frac{\pi}{L} \cdot x\right)$$

$$\left.\begin{matrix} \\ \\ \\ \end{matrix}\right\} \quad \delta\Pi = 0 \quad \Rightarrow \quad P_{cr} = \frac{\pi^2 \cdot EI}{L^2}$$

Exact Euler load because of good Ritz guess

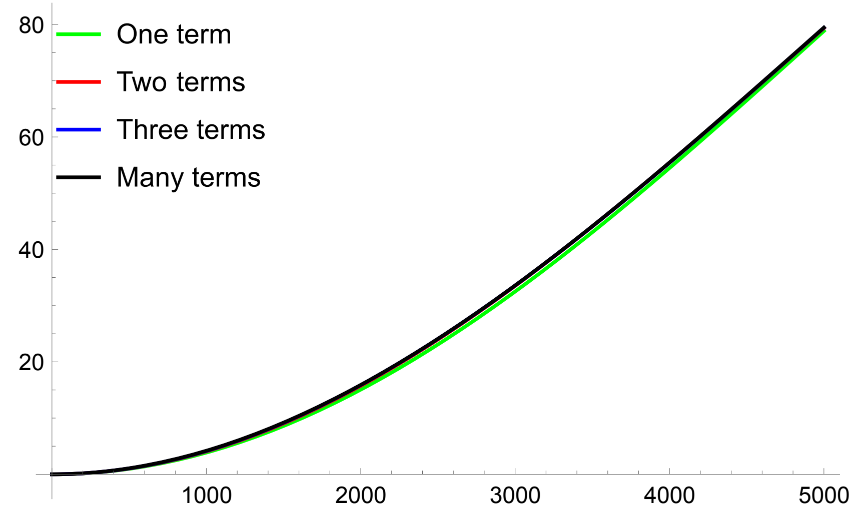
# Cantilever with Axial Force

See example at terje.civil.ubc.ca

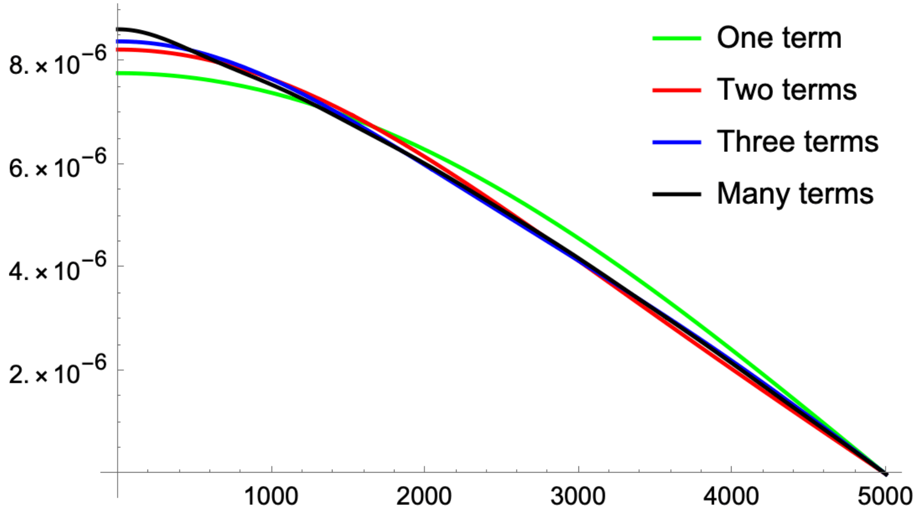



Ritz: $w(x) = \sum a_n \cdot \left(1 - \cos\left(\frac{n \cdot \pi \cdot x}{2\,L}\right)\right)$

Displacement, $w(x)$:




Curvature, $w''(x)$:



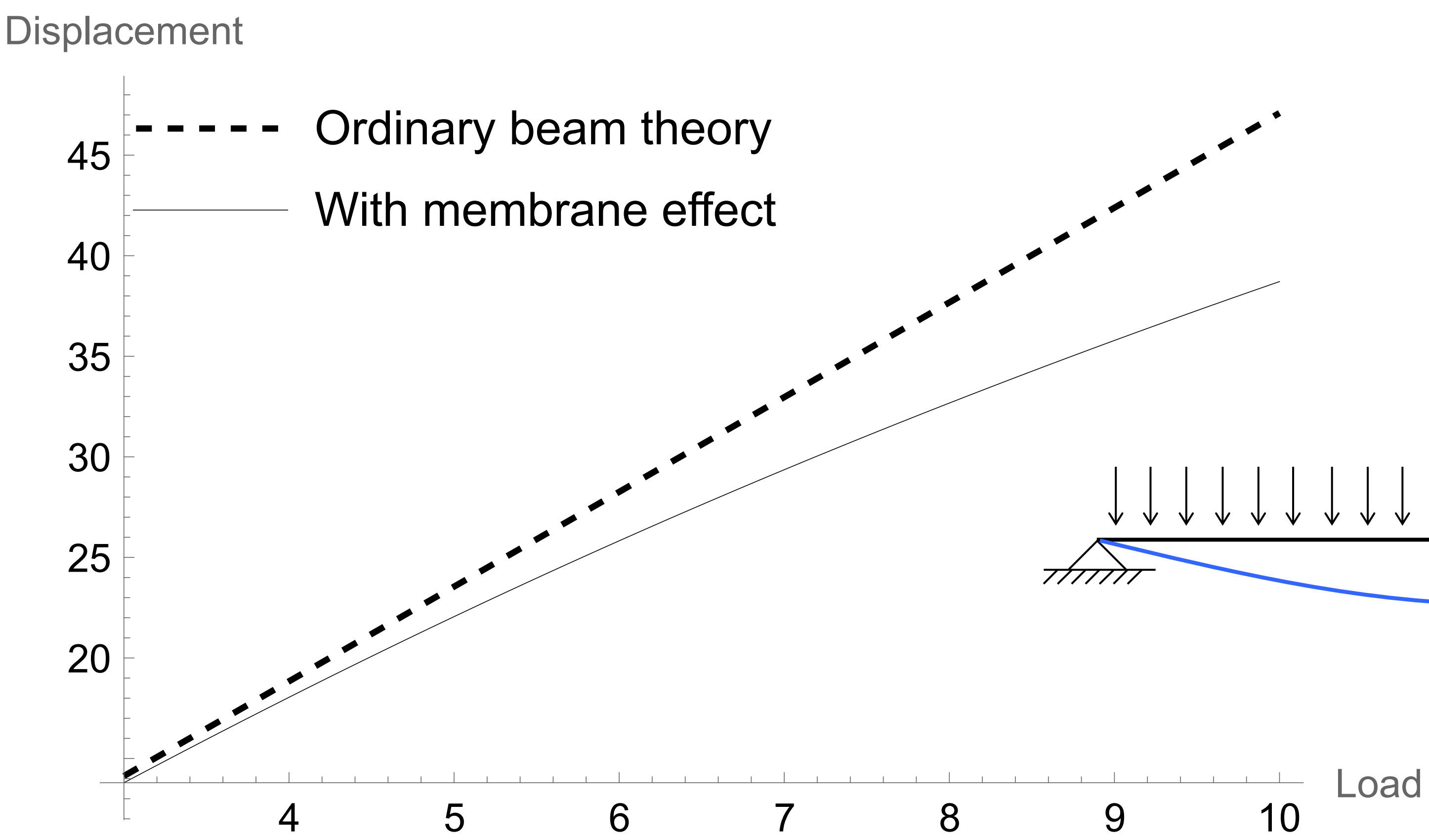
# Membrane Effect

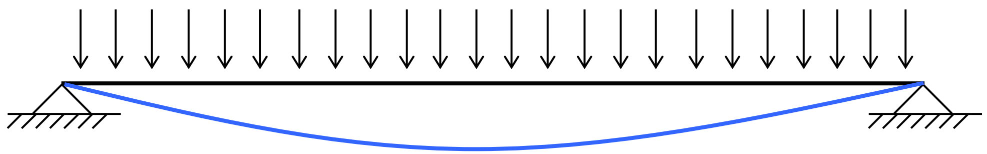
See example at terje.civil.ubc.ca

$$U = \int_V \frac{1}{2} \cdot E \cdot (\varepsilon)^2 \, dV$$

$$\varepsilon = z \cdot \frac{d^2 w}{dx^2} + \frac{1}{2} \cdot \left(\frac{dw}{dx}\right)^2$$

$$H = -\int_0^L q_z \cdot w \, dx$$



More lectures:

Terje’s Toobox:

terje.civil.ubc.ca