A short course on

# Structural Reliability

This lecture:

## Functions & Transformations

Terje's Toolbox is freely available at terje.civil.ubc.ca
It is created and maintained by Professor Terje Haukaas, Ph.D., P.Eng.,
Department of Civil Engineering, The University of British Columbia (UBC), Vancouver, Canada

# Functions vs. Transformations

Tools to understand and solve the invariance problem

**Functions of Random Variables**

Have:

Function $y = h(x)$

Want:

Distribution or (second-moment info) for $y$

**Probability Transformations**

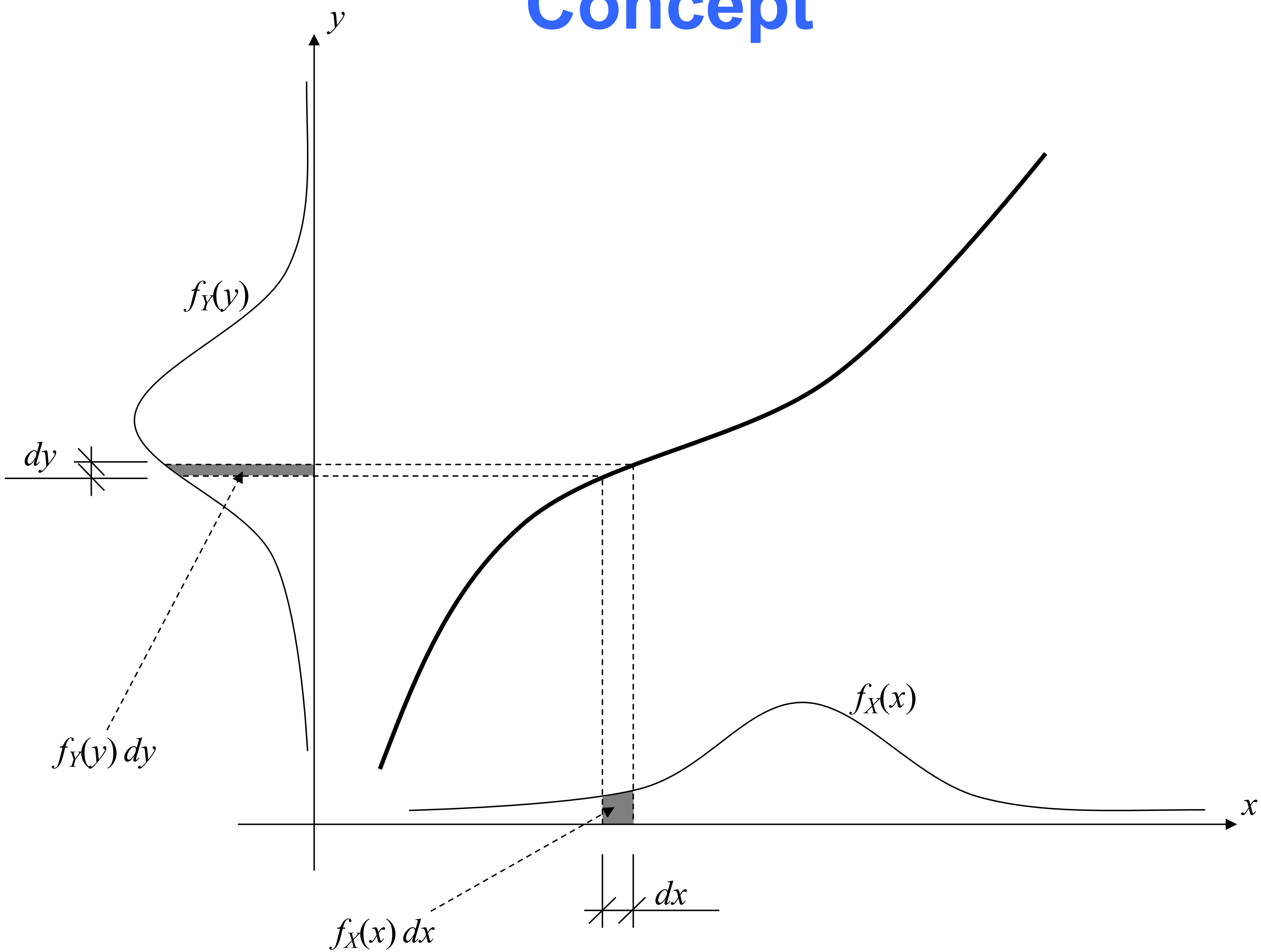
Have:

Distribution (or second-moment info) for $y$

Want:

Functional relationship $y = h(x)$

# Concept



# Derived Distribution

$$Y = h(X)$$

$$X = h^{-1}(Y)$$

$$f_Y(y) \cdot dy = f_X(x) \cdot dx \quad \longleftrightarrow \quad F_Y(y) = F_X(x)$$

$$f_Y(y) = \frac{dx}{dy} \cdot f_X(h^{-1}(y))$$

# Multivariate Case

$$f_{\mathbf{Y}}(y_1, y_2, \cdots, y_n) dy_1 dy_2 \cdots dy_n = f_{\mathbf{X}}(x_1, x_2, \cdots, x_n) dx_1 dx_2 \cdots dx_n$$

$$f_{\mathbf{Y}}(y_1, y_2, \cdots, y_n) = f_{\mathbf{X}}(x_1, x_2, \cdots, x_n) \left| \det(\mathbf{J}_{\mathbf{y},\mathbf{x}}) \right|^{-1}$$

A linear function of **Normal** random variables is **Normal**

A product function of **Lognormal** random variables is **Lognormal**

# Moments: Expectation

$$\mathrm{E}[Y] = \mathrm{E}[h(\mathbf{X})] = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} h(x_1, x_2, \cdots, x_n) f(x_1, x_2, \cdots, x_n)\, dx_1\, dx_2 \cdots dx_n$$

$$\mathrm{E}[a] = a$$

$$\mathrm{E}[a \cdot h(\mathbf{X})] = a \cdot \mathrm{E}[h(\mathbf{X})]$$

$$\mathrm{E}[h_1(\mathbf{X}) + h_2(\mathbf{X})] = \mathrm{E}[h_1(\mathbf{X})] + \mathrm{E}[h_2(\mathbf{X})]$$

$$\frac{\partial}{\partial \theta} \mathrm{E}[h(\mathbf{X}, \theta)] = \mathrm{E}[\frac{\partial}{\partial \theta} h(\mathbf{X}, \theta)]$$

$$\mathrm{Var}[a + X] = \mathrm{Var}[X]$$

$$\mathrm{Var}[b \cdot X] = b^2 \cdot \mathrm{Var}[X]$$

# Linear Functions

$$Y = a + \mathbf{b}^T \mathbf{X} = a + b_i X_i$$

$$\mu_Y = \mathrm{E}[Y] = a + b_i \cdot \mathrm{E}[X_i] = a + \mathbf{b}^T \mathbf{M}_X$$

$$\begin{aligned}
\sigma_Y^2 &= \mathrm{E}\left[\left(Y - \mu_Y\right)^2\right] \\
&= \mathrm{E}\left[\left(\left(a + b_i X_i\right) - \left(a + b_j \mu_j\right)\right)^2\right] \\
&= \mathrm{E}\left[\left(b_i X_i - b_j \mu_j\right)^2\right] \\
&= \mathrm{E}\left[\left(b_i X_i - b_j \mu_j\right) \cdot \left(b_k X_k - b_l \mu_l\right)\right] \\
&= \mathrm{E}\left[b_i X_i \cdot b_k X_k - b_i X_i \cdot b_l \mu_l - b_j \mu_j \cdot b_k X_k + b_j \mu_j \cdot b_l \mu_l\right] \\
&= \mathrm{E}\left[b_i X_i \cdot b_k X_k\right] - \mathrm{E}\left[b_i X_i \cdot b_l \mu_l\right] - \mathrm{E}\left[b_j \mu_j \cdot b_k X_k\right] + \mathrm{E}\left[b_j \mu_j \cdot b_l \mu_l\right] \\
&= b_i b_k \cdot \mathrm{E}\left[X_i X_k\right] - b_j b_l \cdot \mu_j \mu_l \\
&= b_i b_k \cdot \mathrm{Cov}\left[X_i X_k\right] \\
&= \mathbf{b}^T \mathbf{\Sigma}_{XX} \mathbf{b}
\end{aligned}$$

# Correlation

$Y_1 = a + \mathbf{b}^{\mathrm{T}}\mathbf{X}$:

$Y_2 = c + \mathbf{d}^{\mathrm{T}}\mathbf{X}$:

$$\mathrm{Cov}\left[Y_1, Y_2\right] = \mathrm{E}\left[\left(Y_1 - \mu_{Y_1}\right)\left(Y_2 - \mu_{Y_2}\right)\right] = \mathbf{b}^T \mathbf{\Sigma}_{XX}\mathbf{d}$$

$$\rho = \frac{\mathrm{Cov}[Y_1, Y_2]}{\sigma_1 \cdot \sigma_2}$$

Previous slide

# Nonlinear Functions: First-order Approximation

$$Y = h(\mathbf{X}) \approx h(\mathbf{M}_X) + \nabla h(\mathbf{M}_X)^T (\mathbf{X} - \mathbf{M}_X)$$

$$\mu_Y = h(\mathbf{M}_X)$$

$$\sigma_Y^2 = \nabla h(\mathbf{M}_X)^T \mathbf{\Sigma}_{XX} \nabla h(\mathbf{M}_X)$$

$$\mathrm{Cov}[Y_1, Y_2] = \nabla h_1(\mathbf{M}_X)^T \mathbf{\Sigma}_{XX} \nabla h_2(\mathbf{M}_X)$$

# Nonlinear Functions: Second-order Approximation

$$Y = h(\mathbf{X}) \approx h(\mathbf{M_X}) + \frac{\partial h(\mathbf{M_X})}{\partial \mathbf{x}}^{\mathrm{T}} (\mathbf{x} - \mathbf{M_X}) + \frac{1}{2} (\mathbf{x} - \mathbf{M_X})^{\mathrm{T}} \frac{\partial^2 h(\mathbf{M_X})}{\partial \mathbf{x}^2} (\mathbf{x} - \mathbf{M_X})$$

Gradient vector, $\nabla h$

Hessian matrix, $\mathbf{H}$

$$\mu_Y = h(\mathbf{M_X}) + \frac{1}{2} \cdot \mathbf{H} \circ \mathbf{\Sigma} = h(\mathbf{M_X}) + \frac{1}{2} \cdot \sum_{i=1}^{N} \sum_{j=1}^{N} H_{ij} \cdot \Sigma_{ij} = h(\mathbf{M_X}) + \frac{1}{2} \cdot H_{ij} \cdot \Sigma_{ij}$$

Hadamard element-wise multiplication and then summation

# Probability Transformations

Tools to understand and solve the invariance problem

**Functions of Random Variables**

Have:

Function $y = h(x)$

Want:

Distribution or (second-moment info) for $y$

**Probability Transformations**

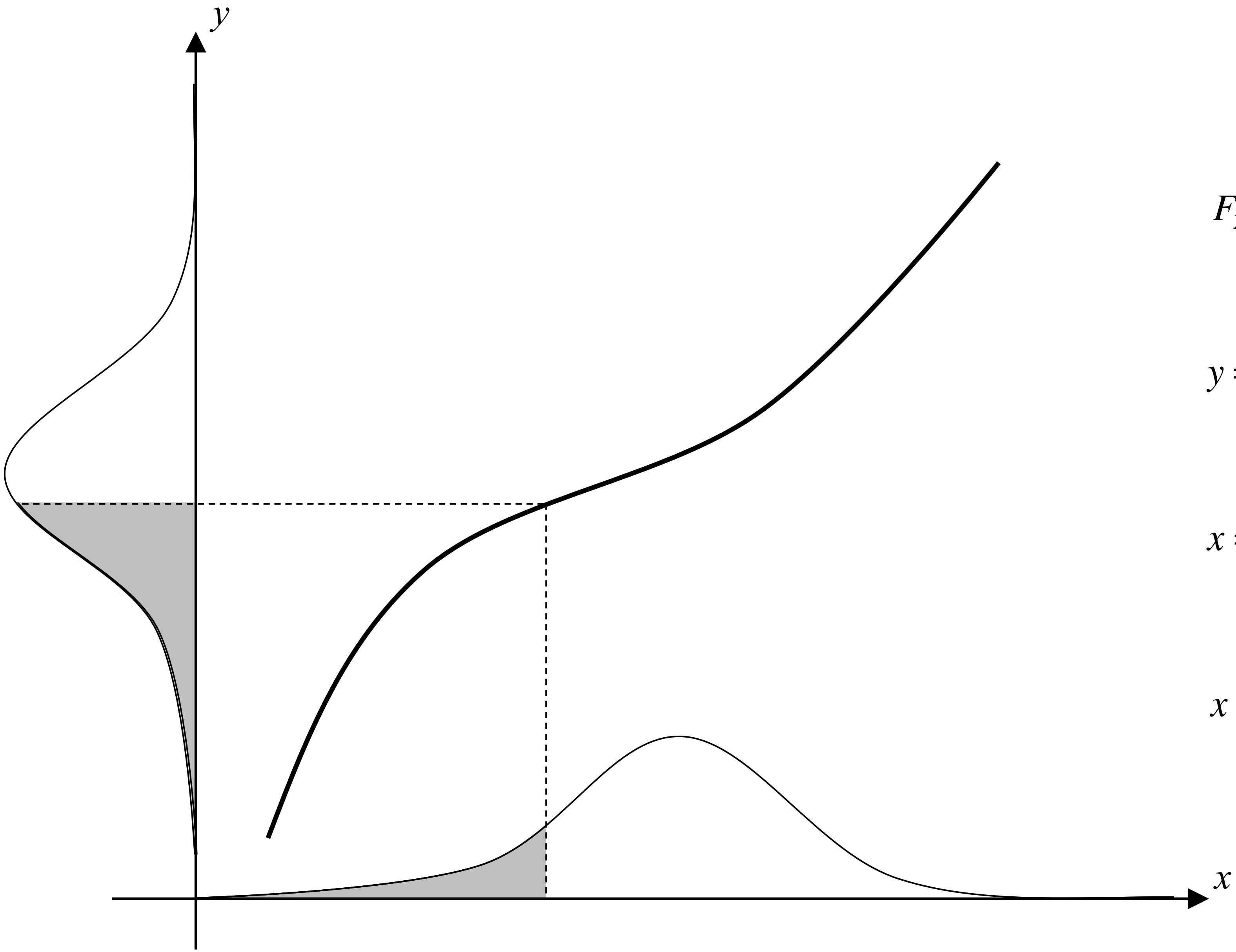
Have:

Distribution (or second-moment info) for $y$

Want:

Functional relationship $y = h(x)$

# Probability-preserving Transformation




$$F_X(x) = F_Y(y)$$

$$y = F_Y^{-1}\left(F_X(x)\right)$$

$$x = F_X^{-1}\left(F_Y(y)\right)$$

$$x = F^{-1}\left(\Phi(y)\right)$$

# Standardize One Variable

Standard means "zero mean and unit variance"

$$y = \frac{x - \mu}{\sigma} \quad \Longleftrightarrow \quad x = \mu + \sigma \cdot y$$

# Standardize Vector

$$\mathbf{y} = \mathbf{a} + \mathbf{B}\mathbf{x}$$

$$\mathbf{M}_Y = \mathbf{a} + \mathbf{B}\mathbf{M}_X = \mathbf{0}$$

$$\mathbf{\Sigma}_{YY} = \mathbf{B}\mathbf{\Sigma}_{XX}\mathbf{B}^T = \mathbf{I}$$

$$\mathbf{\Sigma}_{XX} = \mathbf{B}^{-1}\mathbf{B}^{-T}$$

$$\mathbf{\Sigma}_{XX} = \tilde{\mathbf{L}}\tilde{\mathbf{L}}^T$$

$$\mathbf{B} = \tilde{\mathbf{L}}^{-1}$$

$$\mathbf{a} = -\tilde{\mathbf{L}}^{-1}\mathbf{M}_X$$

$$\mathbf{y} = \tilde{\mathbf{L}}^{-1}\left(\mathbf{x} - \mathbf{M}_X\right)$$

$$\mathbf{x} = \mathbf{M}_X + \tilde{\mathbf{L}}\mathbf{y}$$

# Cholesky Decomposition

```
def choleskyDecomposition(A):

    # Get the size of the input matrix
    n = A.shape[0]

    # Create output matrices of the same size
    L = np.zeros((n, n))
    inverseL = np.zeros((n, n))

    # Loop over rows and columns
    for i in range(n):

        for j in range(n):

            if (i < j):

                L.itemset((i, j), 0.0)

            elif (i == j):

                sum = 0.0

                for k in range(i):

                    sum += L[i, k] * L[i, k]

                if (A[i, j] - sum > 0.0):

                    L.itemset((i, j), np.sqrt(A[i, j] - sum))

                else:
```

Covariance matrix, $\Sigma$: Entries depend on units

Correlation matrix, $\mathbf{R}$: $-1 < \rho < 1$

$$\mathbf{\Sigma}_{XX} = \mathbf{D}_X \mathbf{R}_{XX} \mathbf{D}_X = \mathbf{D}_X \mathbf{L}\mathbf{L}^T \mathbf{D}_X$$

$$\mathbf{y} = \mathbf{L}^{-1}\mathbf{D}_X^{-1}\left(\mathbf{x} - \mathbf{M}_X\right) \quad \Leftrightarrow \quad \mathbf{x} = \mathbf{M}_X + \mathbf{D}_X \mathbf{L}\mathbf{y}$$

# Nataf Transformation

$$z_i = \Phi^{-1}\left(F_i(x_i)\right)$$

$z_i$ are Normal, assume jointly Normal

$$\mathbf{y} = \mathbf{L}^{-1}\mathbf{z} \qquad \Leftrightarrow \qquad \mathbf{z} = \mathbf{L}\mathbf{y}$$

… because $z_i$ are standardized

… but what is the correlation between $z_i$ and $z_j$ ?

Solve from:

$$\rho_{ij} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{x_i - \mu_i}{\sigma_i}\right)\left(\frac{x_j - \mu_j}{\sigma_j}\right)\varphi_2(z_i, z_j, \rho_{0,ij})\, dz_i\, dz_j$$

# Professor Der Kiureghian

"Structural and System Reliability" 2022 textbook (Cambridge Press)

**Table A3.1 Factor $C$ for $X_i$ normal and $X_j$ a standardizable variable (after Liu and Der Kiureghian, 1986)**

| Distribution of $X_j$ | $C$ |
|---|---|
| Uniform | 1.023 |
| Shifted exponential | 1.107 |
| Shifted Rayleigh | 1.014 |
| Type I largest value | 1.031 |
| Type I smallest value | 1.031 |

**Table A3.2 Factor $C$ for $X_i$ normal and $X_j$ a non-standardizable variable (after Liu and Der Kiureghian, 1986)**

| Distribution of $X_j$ | $C$ |
|---|---|
| Lognormal | $\frac{\delta_j}{\sqrt{\ln(1+\delta_j^2)}}$ |
| Gamma | $1.001 - 0.007\delta_j + 0.118\delta_j^2$ |
| Type II largest value | $1.030 + 0.238\delta_j + 0.364\delta_j^2$ |
| Type III smallest value | $1.031 - 0.195\delta_j + 0.328\delta_j^2$ |

**Table A3.3 Factor $C$ for $X_i$ and $X_j$ both standardizable variables (after Liu and Der Kiureghian, 1986)**

| Distribution of $X_i$ | Distribution of $X_j$: Uniform | Shifted exponential | Shifted Rayleigh | Type I largest value | Type I smallest value |
|---|---|---|---|---|---|
| Uniform | $1.047 - 0.047\rho^2$ | | | | |
| Shifted exponential | $1.133 + 0.029\rho^2$ | $1.229 - 0.367\rho + 0.153\rho^2$ | | | |
| Shifted Rayleigh | $1.038 - 0.008\rho^2$ | $1.123 - 0.100\rho + 0.021\rho^2$ | $1.028 - 0.029\rho$ | | |
| Type I largest value | $1.055 + 0.015\rho^2$ | $1.142 - 0.154\rho + 0.031\rho^2$ | $1.046 - 0.045\rho + 0.006\rho^2$ | $1.064 - 0.069\rho + 0.005\rho^2$ | |
| Type I smallest value | $1.055 + 0.015\rho^2$ | $1.142 + 0.154\rho + 0.031\rho^2$ | $1.046 + 0.045\rho + 0.006\rho^2$ | $1.064 + 0.069\rho + 0.005\rho^2$ | $1.064 - 0.069\rho + 0.005\rho^2$ |

More lectures:

Terje’s Toobox:

terje.civil.ubc.ca