A short course on

Structural Reliability

This lecture:

FOSM & FORM

Terje's Toolbox is freely available at terje.civil.ubc.ca
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Reliability Methods

Reliability = $1 - p_f$

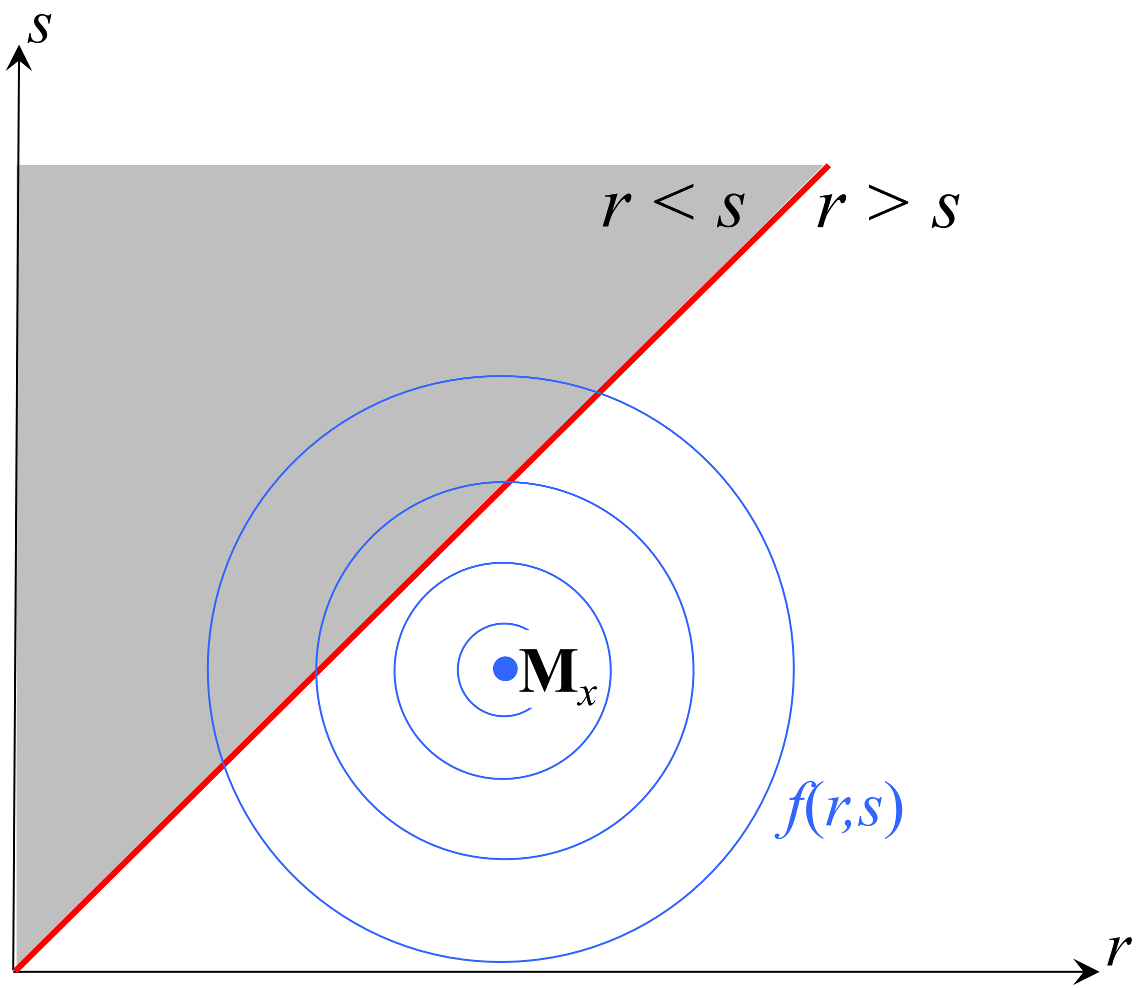
$$p_f = \mathrm{P}(g \leq 0) = \int_{g \leq 0} \cdots \int f(\mathbf{x})\, d\mathbf{x}$$

MCFOSM	= mean-centred first-order second-moment method
FOSM	= first-order second-moment method
FORM	= first-order reliability method
SORM	= second-order reliability method
MC	= Monte Carlo sampling
IS	= importance sampling
SS	= subset sampling
DS	= directional sampling

MCFOSM

$$p_f = \mathrm{P}(g \le 0) = \Phi\left(\frac{g-\mu_g}{\sigma_g}\right) = \Phi\left(-\frac{\mu_g}{\sigma_g}\right) = \Phi(-\beta)$$

(Why seek p_f? Appreciate β !)

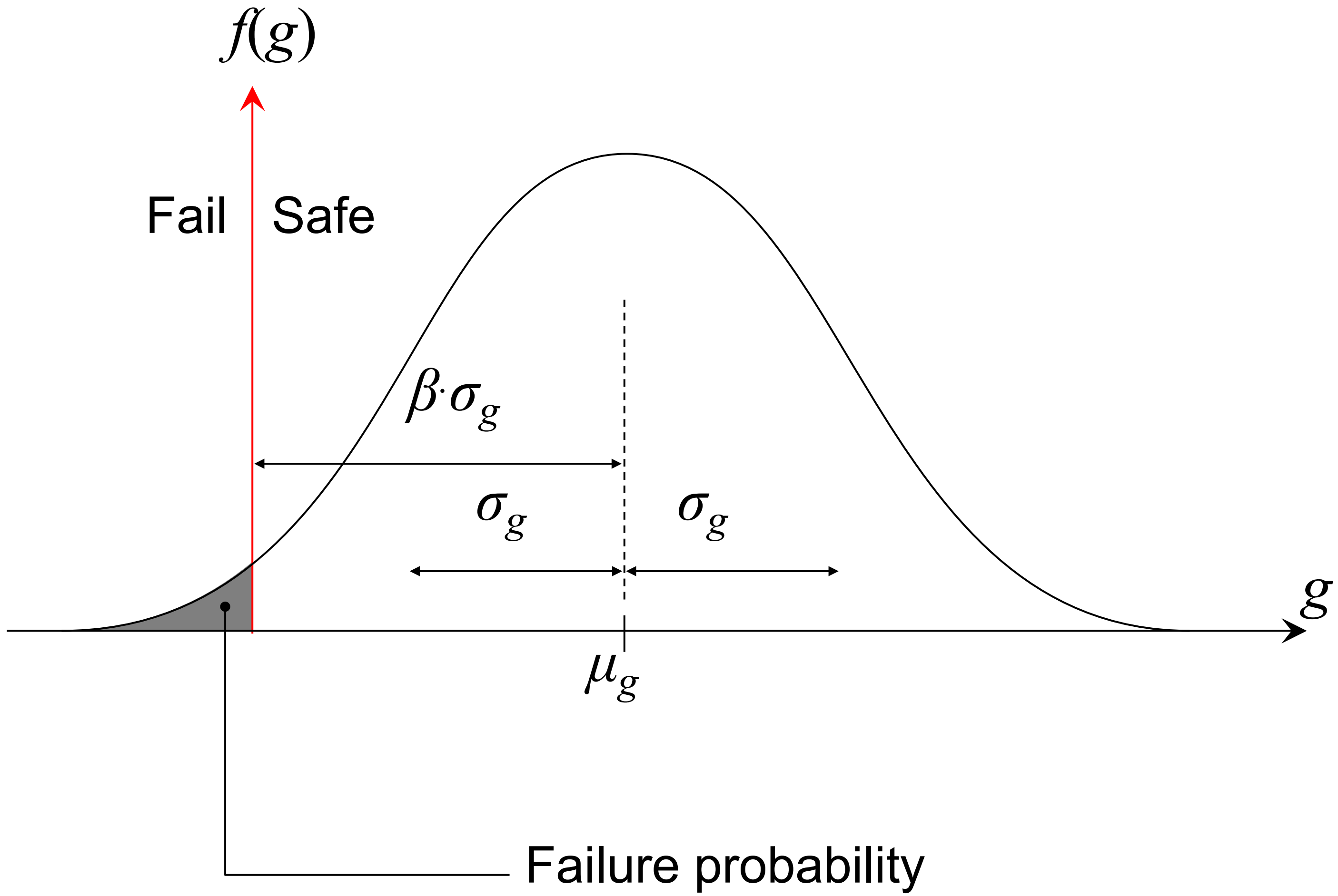


$$\beta = \frac{\mu_g}{\sigma_g}$$

$$\mu_g \approx g(\mathbf{M}_X)$$

$$\sigma_g \approx \sqrt{\nabla g(\mathbf{M}_X)^T \mathbf{\Sigma}_{XX} \nabla g(\mathbf{M}_X)}$$

What is β ?



$f(g)$
Fail
Safe
$\beta \cdot \sigma_g$
σ_g
σ_g
g
μ_g
Failure probability

Invariance Problem

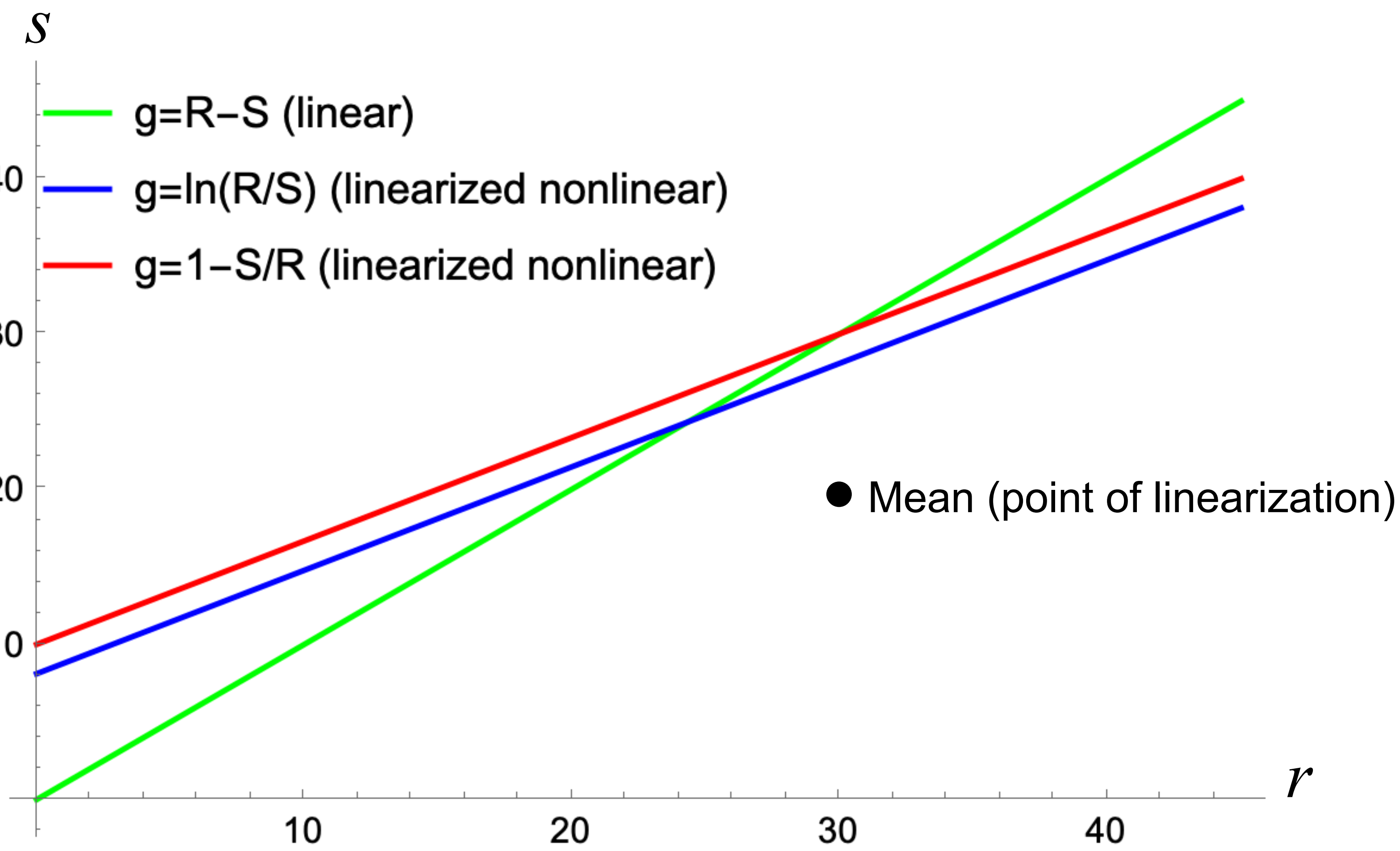
μ_R=30, μ_S=20, σ_R=5, σ_S=10, ρ_{RS}=0.5

MCFOSM:

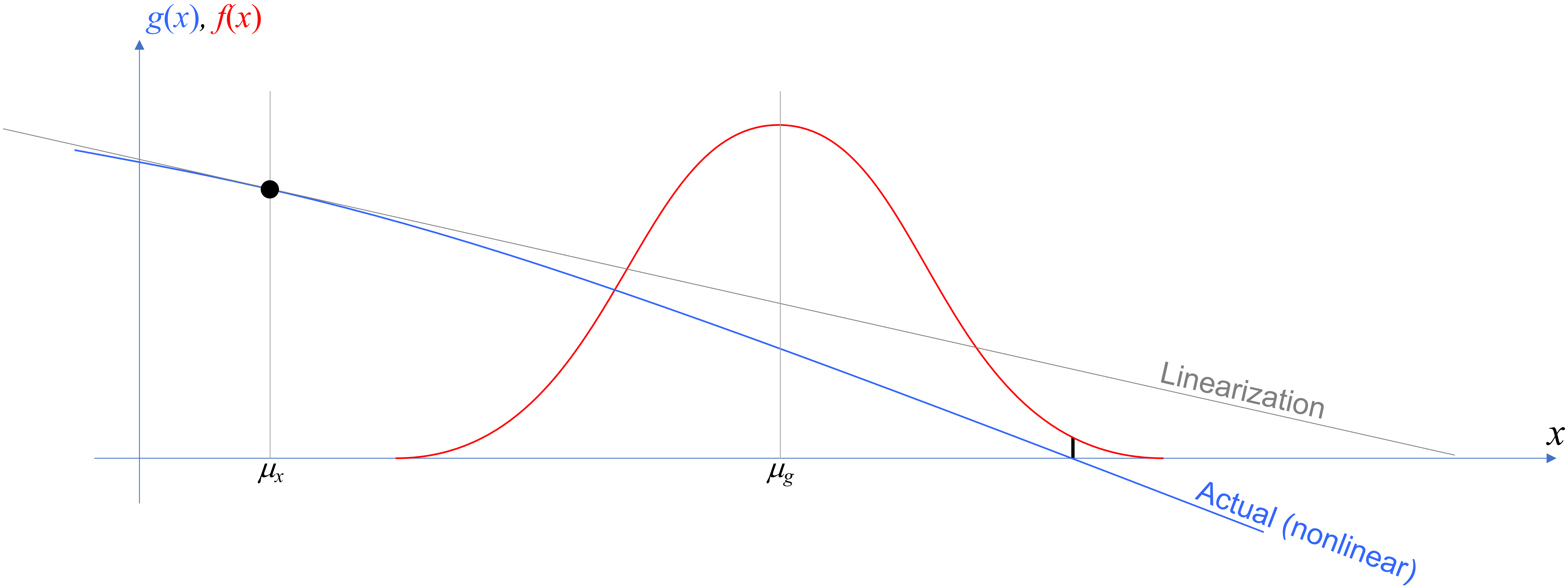
$\beta_1 = 1.15$

$\beta_2 = 0.92$

$\beta_3 = 1.13$



Univariate Explanation



Solution

Linearizing nonlinear $g(\mathbf{x})$ at $\mathbf{x}=\mathbf{M}_x$ causes the invariance problem

FOSM and FORM linearizes somewhere on the limit-state surface

To understand & solve the issue:

Functions of random variables

Probability transformations

Geometric Interpretation of β

Consider a linear limit-state function: $g(\mathbf{X}) = a + \mathbf{b}^T\mathbf{X}$

Transform into a standardized space: $\mathbf{X} = \mathbf{M}_X + \mathbf{D}_X\mathbf{L}\mathbf{Y}$

Resulting limit-state function (notice capital G):

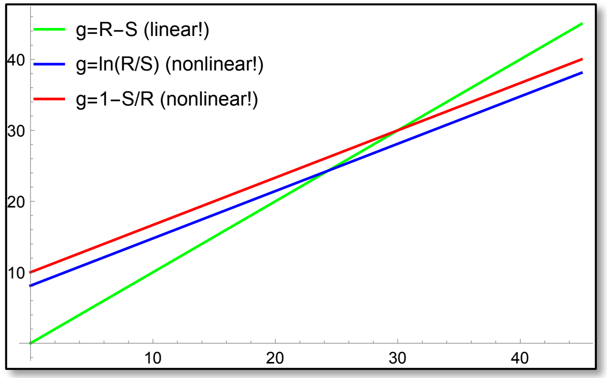
$$G(\mathbf{Y}) = a + \mathbf{b}^T\mathbf{M}_X + \mathbf{b}^T\mathbf{D}_X\mathbf{L}\mathbf{Y}$$
$$= c + \mathbf{d}^T\mathbf{Y}$$

MCFOSM: $$\beta = \frac{\mu_G}{\sigma_G} = \frac{c}{\sqrt{\mathbf{d}^T\mathbf{d}}} = \frac{c}{\|\mathbf{d}\|}$$

Identical!

Distance from point to plane: $$\Delta = \left|\frac{G(\mathbf{0})}{\|\nabla G\|}\right| = \left|\frac{c}{\|\mathbf{d}\|}\right|$$

Original space:



Visualization

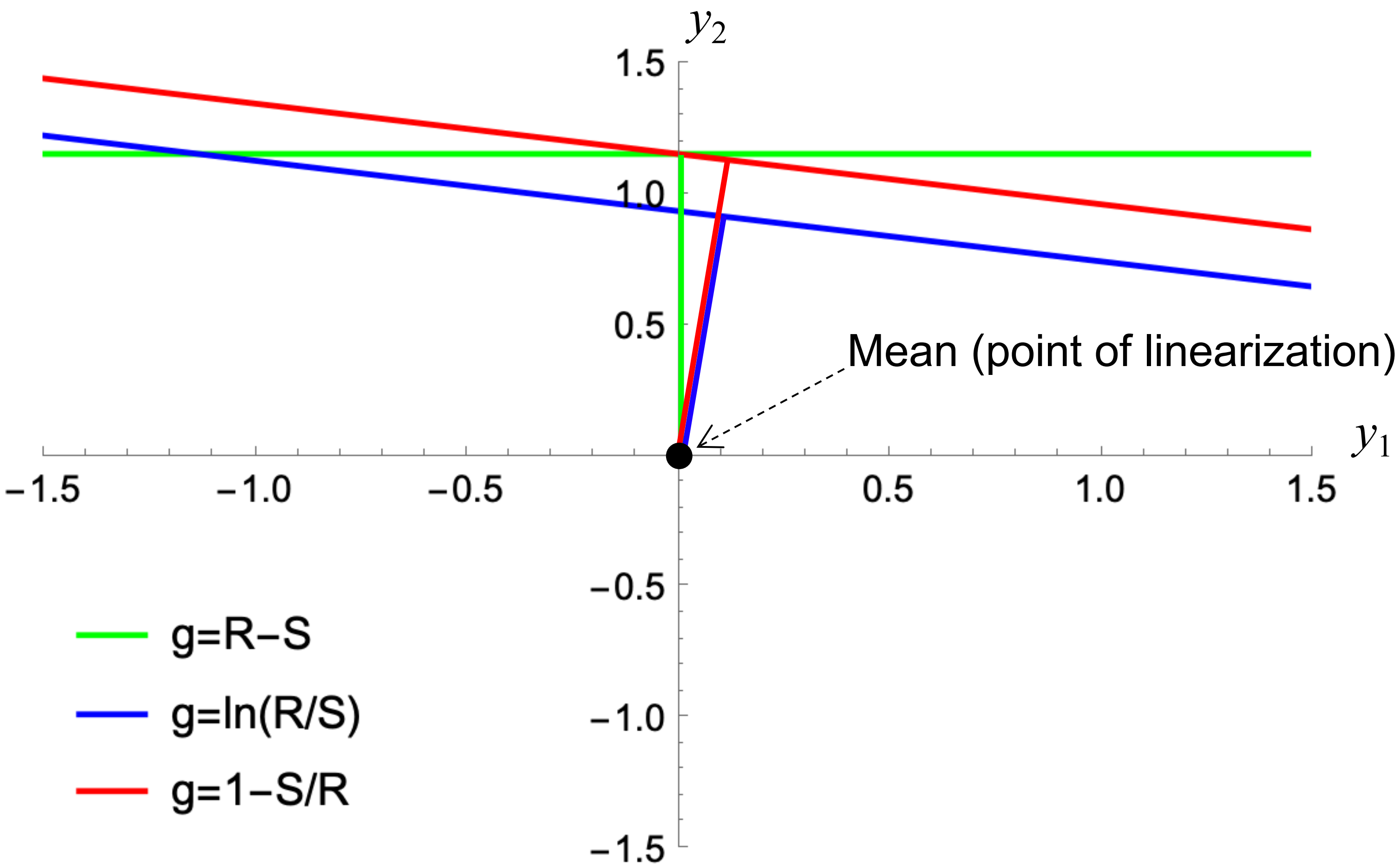
μ_R=30, μ_S=20, σ_R=5, σ_S=10, ρ_{RS}=0.5

MCFOSM:

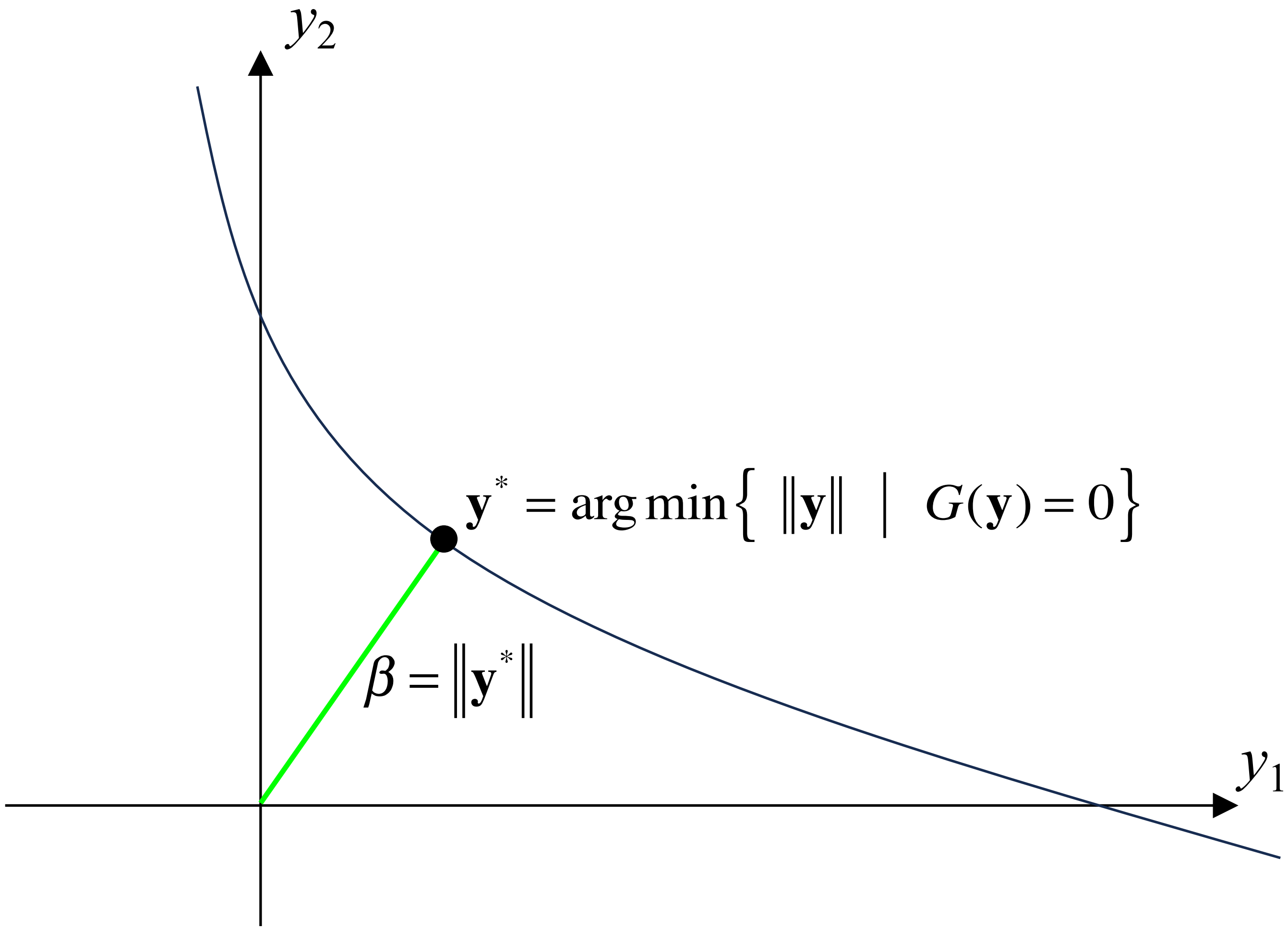
$\beta_1 = 1.15$

$\beta_2 = 0.92$

$\beta_3 = 1.13$



Design Point



Sequential Linearization

$$G(\mathbf{y}) \approx G(\mathbf{y}_m) + \nabla G(\mathbf{y}_m)^T \cdot (\mathbf{y} - \mathbf{y}_m) = 0$$

$$\Delta = \frac{G(\mathbf{0})}{\left\|\nabla G(\mathbf{y}_m)\right\|} = \frac{G(\mathbf{y}_m) - \nabla G(\mathbf{y}_m)^T \cdot \mathbf{y}_m}{\left\|\nabla G(\mathbf{y}_m)\right\|} = \frac{G(\mathbf{y}_m)}{\left\|\nabla G(\mathbf{y}_m)\right\|} + \boldsymbol{\alpha}^T \cdot \mathbf{y}_m$$

$$\mathbf{y}_{m+1} = -\Delta \cdot \frac{\nabla G(\mathbf{y}_m)}{\left\|\nabla G(\mathbf{y}_m)\right\|} \equiv \Delta \cdot \boldsymbol{\alpha}$$

Search Direction & Step Size

$$\mathbf{y}_{m+1} = \mathbf{y}_m + s_m \cdot \mathbf{d}_m$$

$$\mathbf{d}_m = \mathbf{y}_{m+1} - \mathbf{y}_m = \left(\frac{G(\mathbf{y}_m)}{\left\| \nabla G(\mathbf{y}_m) \right\|} + \boldsymbol{\alpha}^T \cdot \mathbf{y}_m \right) \boldsymbol{\alpha} - \mathbf{y}_m$$

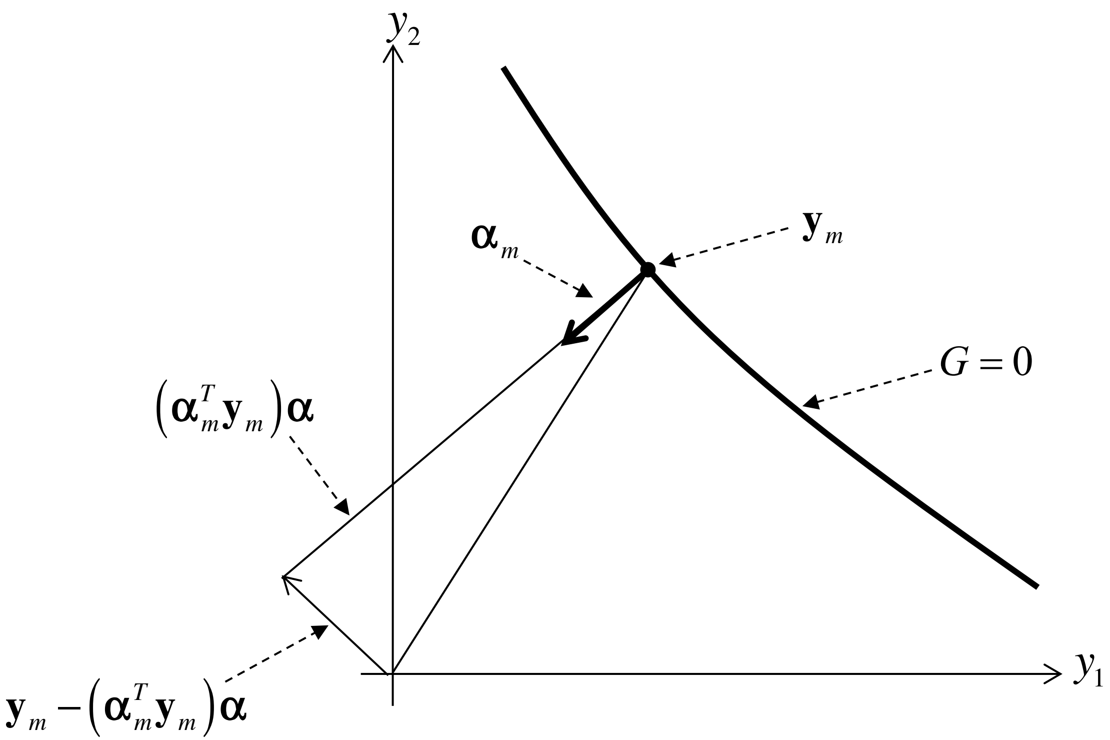
Step size options:

- $s_m = 1$
- $s_m = b^k$ (Armijo's rule, typically b=0.5, k=0,1,2,3,…)
- s_m = Unidirectional optimization algorithm (golden section, etc.)

Convergence Criteria

$$\left| \frac{G(\mathbf{y}_m)}{G_0} \right| \leq e_1$$

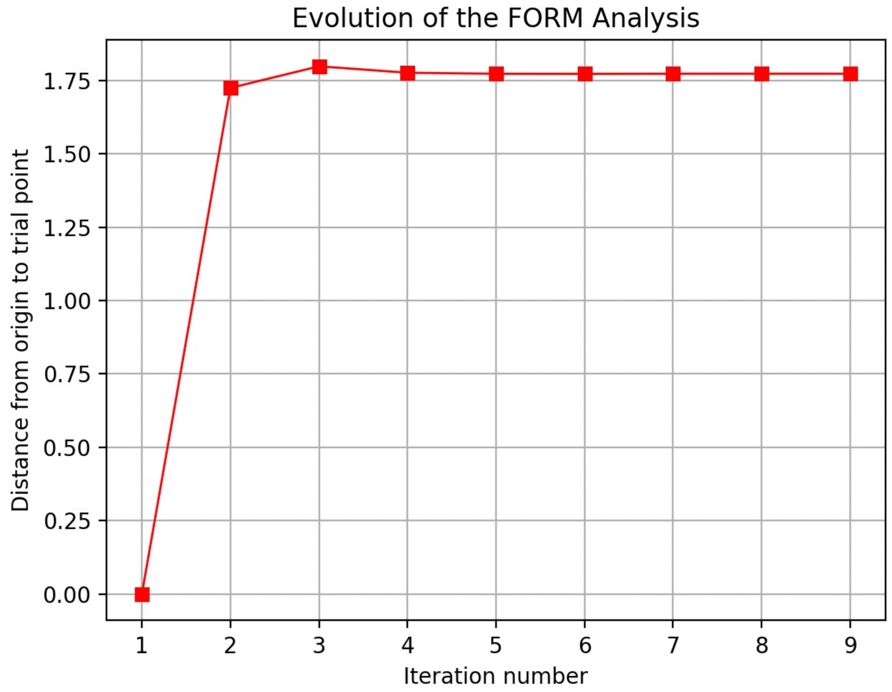
$$\left\| \mathbf{y}_m - \left(\boldsymbol{\alpha}_m^T \mathbf{y}_m \right) \boldsymbol{\alpha}_m \right\| \leq e_2$$

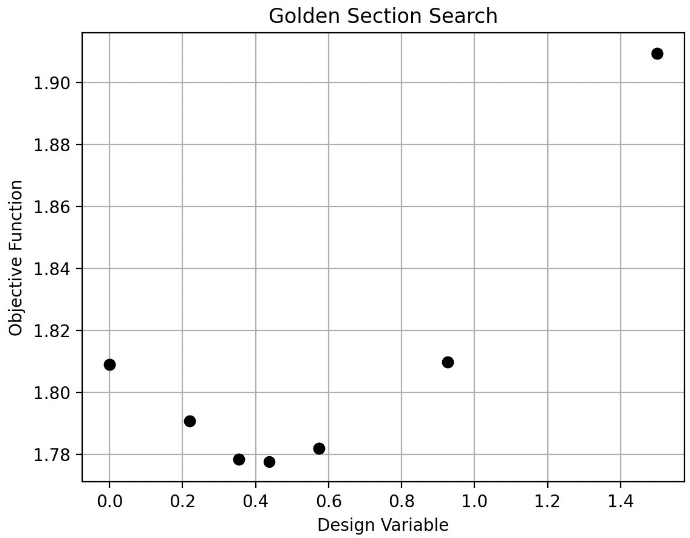


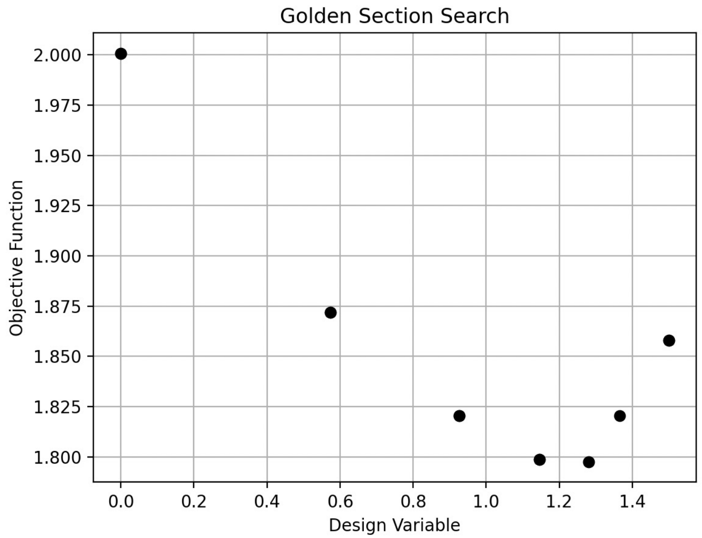
Analysis Procedure

1. Set $m=1$
2. Select starting point in standard space: $\mathbf{y}_m$
3. Transform into original space: $\mathbf{y}_m \rightarrow \mathbf{x}_m$
4. Evaluate limit-state function $G(\mathbf{y}_m)=g(\mathbf{x}_m)$
5. Evaluate gradient of limit-state function: $\frac{\partial G}{\partial \mathbf{y}} = \frac{\partial g}{\partial \mathbf{u}} \frac{\partial \mathbf{u}}{\partial \mathbf{x}} \frac{\partial \mathbf{x}}{\partial \mathbf{y}}$
6. If $m=1$, set scaling factor for first convergence criterion: $G_0=G(\mathbf{y}_m)$
7. Check convergence: $\left| \frac{G(\mathbf{y}_m)}{G_0} \right| \leq e_1$ $\quad \left\| \mathbf{y}_m - \left(\boldsymbol{\alpha}_m^T \mathbf{y}_m \right) \boldsymbol{\alpha}_m \right\| \leq e_2$
8. If convergence is NOT achieved, go back to Step 3 with: $\mathbf{y}_{m+1} = \mathbf{y}_m + s_m \cdot \mathbf{d}_m$
9. If convergence is achieved: $\beta = \left\| \mathbf{y}^* \right\|$ $\quad p_f = \Phi(-\beta)$

Evolution of the Search







More slides to come here…

More lectures:

Terje's Toobox:

terje.civil.ubc.ca