Nonlinear Dynamic Analysis

The governing system of equations for this class of problems is

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{C}\dot{\mathbf{u}} + \tilde{\mathbf{F}}(\mathbf{u}) = \mathbf{F} \tag{1}$$

where **M**=mass matrix, **u**=displacement vector, each dot means one derivative with respect to time, **C**=damping matrix, $\tilde{\mathbf{F}}$=internal resisting force, and **F**=applied forces. As in the document on linear dynamic problems, Rayleigh damping is assumed, implying that **C** is a combination of **M** and **K**. By introducing the generic time-stepping scheme

$$\begin{aligned} \ddot{\mathbf{u}}_{n+1} &= a_1 \cdot \mathbf{u}_{n+1} + a_2 \cdot \mathbf{u}_n + a_3 \cdot \dot{\mathbf{u}}_n + a_4 \cdot \ddot{\mathbf{u}}_n \\ \dot{\mathbf{u}}_{n+1} &= a_5 \cdot \mathbf{u}_{n+1} + a_6 \cdot \mathbf{u}_n + a_7 \cdot \dot{\mathbf{u}}_n + a_8 \cdot \ddot{\mathbf{u}}_n \end{aligned} \tag{2}$$

where the special case of the Newmark algorithm with β=0.25 and γ=0.5 is written

- $a_1 = 1/(\beta\,\Delta t^2)$
- $a_2 = -a_1$
- $a_3 = -1.0/(\beta\,\Delta t)$
- $a_4 = 1.0-1/(2\beta)$
- $a_5 = \gamma/(\beta\,\Delta t)$
- $a_6 = -a_5$
- $a_7 = 1-\gamma/\beta$
- $a_8 = \Delta t\,(1-\gamma/(2\beta))$

the space- and time-discretized equations of motion read

$$\mathbf{M}(a_1\mathbf{u}_{n+1} + a_2\mathbf{u}_n + a_3\dot{\mathbf{u}}_n + a_4\ddot{\mathbf{u}}_n) + \mathbf{C}(a_5\mathbf{u}_{n+1} + a_6\mathbf{u}_n + a_7\dot{\mathbf{u}}_n + a_8\ddot{\mathbf{u}}_n) + \tilde{\mathbf{F}}_{n+1} = \mathbf{F}_{n+1} \tag{3}$$

In the context of nonlinear analysis, Eq. (3) may be compared to $\tilde{\mathbf{F}} = \mathbf{F}$ for static analysis. In static analysis, the residual, $\mathbf{R} = \tilde{\mathbf{F}} - \mathbf{F}$, is defined, and the Newton-Raphson algorithm is essentially $\partial\mathbf{R}/\partial\mathbf{u}\,\Delta\mathbf{u} = -\mathbf{R}$, where the stiffness is $\partial\mathbf{R}/\partial\mathbf{u} = \partial\tilde{\mathbf{F}}/\partial\mathbf{u}$. Similarly, the residual for nonlinear dynamic is

$$\begin{aligned} \mathbf{R} &= \tilde{\mathbf{F}}_{n+1} - \mathbf{F}_{n+1} \\ &+\mathbf{M}(a_1\mathbf{u}_{n+1} + a_2\mathbf{u}_n + a_3\dot{\mathbf{u}}_n + a_4\ddot{\mathbf{u}}_n) + \mathbf{C}(a_5\mathbf{u}_{n+1} + a_6\mathbf{u}_n + a_7\dot{\mathbf{u}}_n + a_8\ddot{\mathbf{u}}_n) \end{aligned} \tag{4}$$

In this case, the extended expression for the stiffness is

$$\frac{\partial\mathbf{R}}{\partial\mathbf{u}_{n+1}} = \frac{\partial\tilde{\mathbf{F}}_{n+1}}{\partial\mathbf{u}_{n+1}} + a_1\mathbf{M} + a_5\mathbf{C} \tag{5}$$

With those new expressions for the residual and its derivative, the iterations of the Newton-Raphson algorithm proceeds, at each time step, as it does for static problems:

Time step *increments*:

Determine load vector, $\mathbf{F}$, e.g., $\mathbf{F} = -\mathbf{\Gamma} \cdot \mathbf{M} \cdot \ddot{u}_g(t)$ for ground motion

Newton-Raphson equilibrium *iterations*:

Calculate effective tangent stiffness $\partial \mathbf{R}/\partial \mathbf{u} = a_1 \mathbf{M} + a_5 \mathbf{C} + \partial \tilde{\mathbf{F}}/\partial \mathbf{u}$

Solve linear system $\partial \mathbf{R}/\partial \mathbf{u}\, \Delta \mathbf{u} = -\mathbf{R}$ for displacement increment, $\Delta \mathbf{u}$

Calculate new trial displacements $\mathbf{u}_{i+1} = \mathbf{u}_i + \Delta \mathbf{u}$

Calculate velocities and accelerations from Eq. (2)

Conduct state determination, i.e., calculate $\tilde{\mathbf{F}}(\mathbf{u}_{n+1})$ and $\partial \tilde{\mathbf{F}}/\partial \mathbf{u}_{n+1}$

Evaluate the residual in Eq. (4)

Keep iterating until the norm of the residual, $||\mathbf{R}||$, is sufficiently small

Upon convergence of iterations, issue commit order to all hysteretic materials