A short course on

Nonlinear Finite Element Analysis

This video:

Newton-Raphson and Load Incrementation

Terje's Toolbox is freely available at terje.civil.ubc.ca
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Notation from Linear Analysis

Equilibrium equations: $\mathbf{Ku} = \mathbf{F}$

Index notation: $K_{ij}\, u_j = F_i$

K_{ij} = force along DOF i due to a unit disp./rot. along DOF j

u_j = unknown displacement or rotation along DOF j

F_i = force along DOF i due to external loads

Split member forces and point loads: $\mathbf{Ku} + \bar{\mathbf{F}} = \grave{\mathbf{F}}$

Total load vector: $\mathbf{Ku} = \grave{\mathbf{F}} - \bar{\mathbf{F}} = \mathbf{F}$

Member end forces after solving equilibrium equations: $\mathbf{F} = \mathbf{Ku} + \bar{\mathbf{F}}$

Nonlinear Analysis

Equilibrium equations: $\tilde{\mathbf{F}}(\mathbf{u}) = \mathbf{F}$

Internal resisting forces (nonlinear function of $\mathbf{u}$)

Externally applied loads

Configurations

$$\tilde{\mathbf{F}}_f \quad — \quad \mathbf{K}_f = \mathbf{T}_{af}^{\mathrm{T}}\mathbf{K}_a\mathbf{T}_{af} \quad — \quad \mathbf{u}_f$$

$$\mathbf{F}_f = \mathbf{T}_{af}^{\mathrm{T}}\mathbf{F}_a \qquad \mathbf{u}_a = \mathbf{T}_{af}\mathbf{u}_f$$

$$\tilde{\mathbf{F}}_a \quad — \quad \mathbf{K}_a = \Sigma\left(\mathbf{T}_{ga}^{\mathrm{T}}\mathbf{K}_g\mathbf{T}_{ga}\right) \quad — \quad \mathbf{u}_a$$

$$\mathbf{F}_a = \Sigma\left(\mathbf{T}_{ga}^{\mathrm{T}}\mathbf{F}_g\right) \qquad \mathbf{u}_g = \mathbf{T}_{ga}\mathbf{u}_a$$

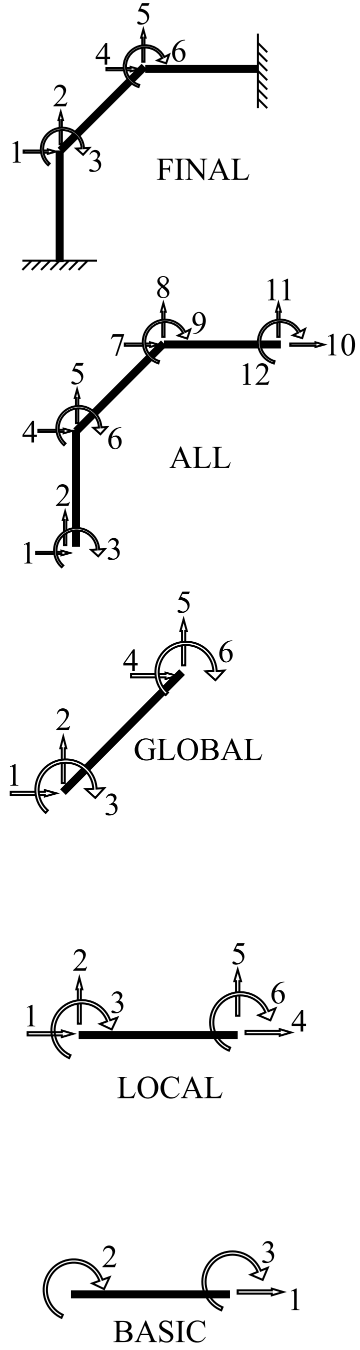
$$\tilde{\mathbf{F}}_g \quad — \quad \mathbf{K}_g = \mathbf{T}_{lg}^{\mathrm{T}}\mathbf{K}_l\mathbf{T}_{lg} \quad — \quad \mathbf{u}_g$$

$$\mathbf{F}_g = \mathbf{T}_{lg}^{\mathrm{T}}\mathbf{F}_l \qquad \mathbf{u}_l = \mathbf{T}_{lg}\mathbf{u}_g$$

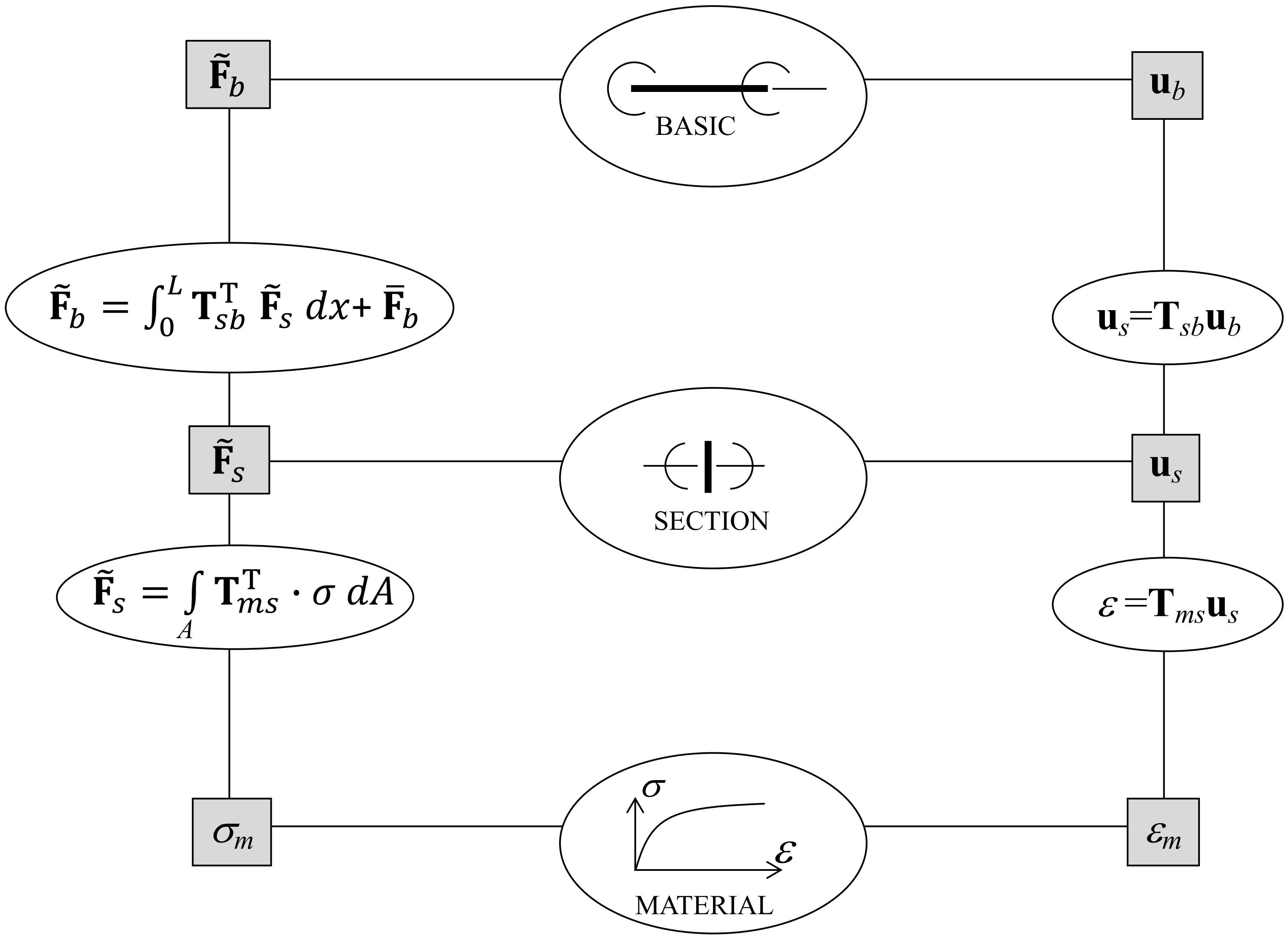
$$\tilde{\mathbf{F}}_l \quad — \quad \mathbf{K}_l = \mathbf{T}_{bl}^{\mathrm{T}}\mathbf{K}_b\mathbf{T}_{bl} \quad — \quad \mathbf{u}_l$$

$$\mathbf{F}_l = \mathbf{T}_{bl}^{\mathrm{T}}\mathbf{F}_b \qquad \mathbf{u}_b = \mathbf{T}_{bl}\mathbf{u}_l$$

$$\tilde{\mathbf{F}}_b \quad — \quad \mathbf{K}_b \quad — \quad \mathbf{u}_b$$



Added in Nonlinear Analysis



Analysis Procedure

1. Determine trial displacements, $\mathbf{u}_f$ (will soon see how)

2. Determine corresponding strain, $\varepsilon = \mathbf{T}_{ms}\mathbf{T}_{sb}\mathbf{T}_{bl}\mathbf{T}_{lg}\mathbf{T}_{ga}\mathbf{T}_{af}\mathbf{u}_f$ (can do better than $\mathbf{T}_{ga}\mathbf{T}_{af}$)

3. Determine stress for given strain from the material law, often history-dependent, i.e., hysteretic

4. Determine resisting forces: $\tilde{\mathbf{F}}_f(\mathbf{u}_f) = \mathbf{T}_{af}^T \sum \left(\mathbf{T}_{ga}^T \mathbf{T}_{lg}^T \mathbf{T}_{bl}^T \left(\int_0^L \mathbf{T}_{sb}^{\mathrm{T}} \left(\int \mathbf{T}_{ms}^{\mathrm{T}} \cdot \sigma \, dA \right) dx + \bar{\mathbf{F}}_b \right) \right)$

5. Check convergence: $\tilde{\mathbf{F}} = \mathbf{F}$?

Newton-Raphson

Equilibrium on residual form: $\tilde{\mathbf{F}}(\mathbf{u}) - \mathbf{F} = \mathbf{R} = \mathbf{0}$

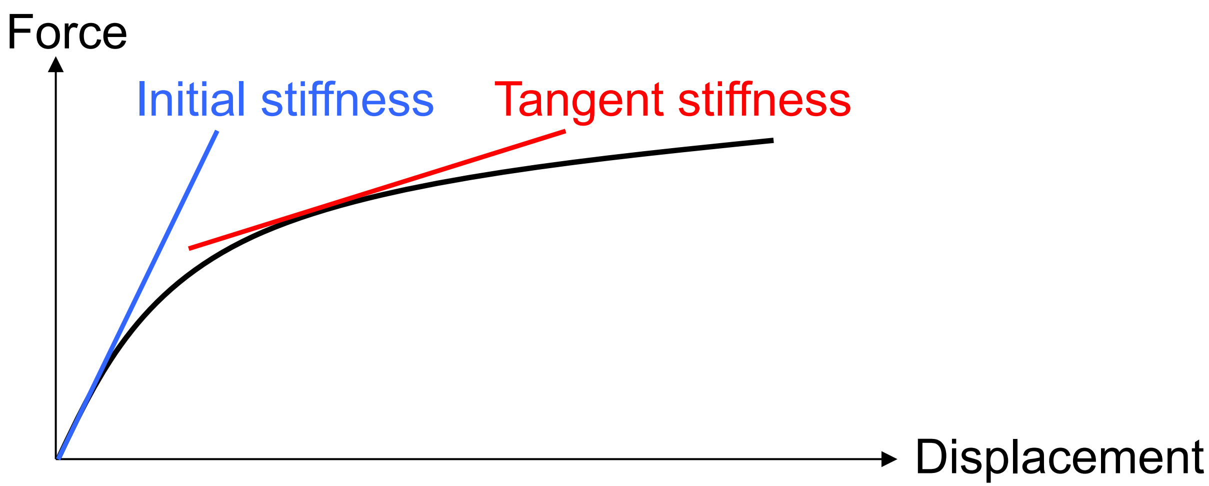
First-order Taylor expansion of the residual: $\mathbf{R}(\mathbf{u}) = \mathbf{R}(\mathbf{u}_i) + \frac{\partial \mathbf{R}(\mathbf{u}_i)}{\partial \mathbf{u}}(\mathbf{u} - \mathbf{u}_i)$

Set the residual to zero and recognize linear system of equations: $\frac{\partial \mathbf{R}(\mathbf{u}_i)}{\partial \mathbf{u}}\Delta\mathbf{u} = -\mathbf{R}(\mathbf{u}_i)$

Because **R** is a nonlinear function of $\mathbf{u}$, we iterate: $\mathbf{u}_{i+1} = \mathbf{u}_i + \Delta\mathbf{u}$ (the trial displacements)

About the derivative: $\frac{\partial \mathbf{R}(\mathbf{u}_i)}{\partial \mathbf{u}} = \frac{\partial \tilde{\mathbf{F}}(\mathbf{u}_i)}{\partial \mathbf{u}} = \mathbf{K}_{\text{tangent}}$

Tangent Stiffness



Modified Newton-Raphson

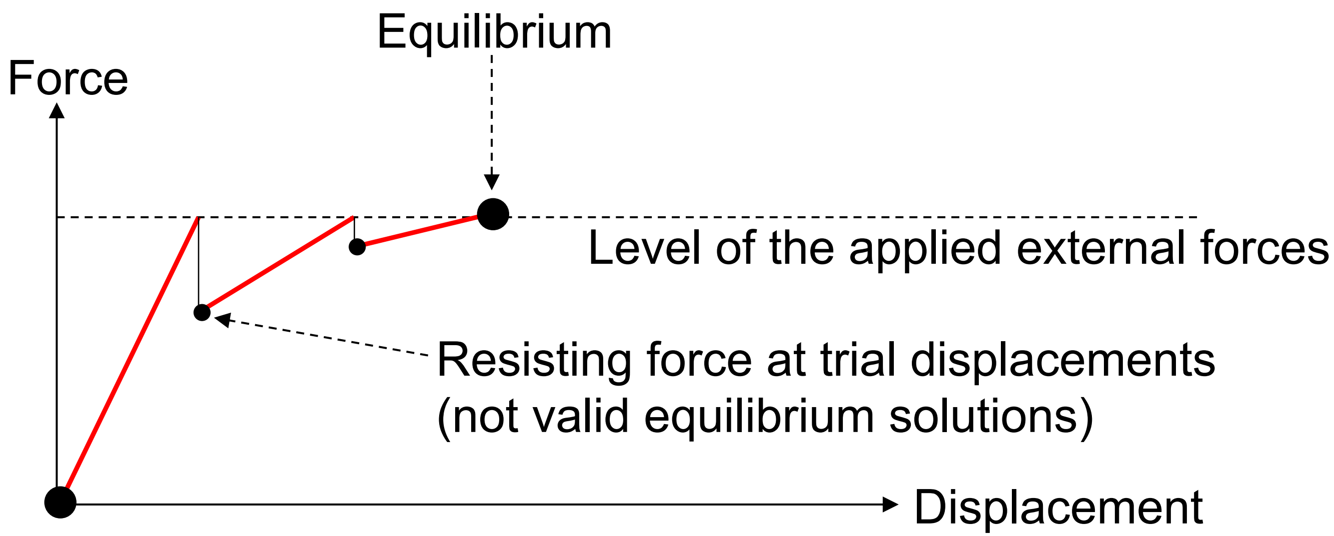
Linear system of equations in each iteration: $\frac{\partial \mathbf{R}(\mathbf{u}_i)}{\partial \mathbf{u}} \Delta \mathbf{u} = -\mathbf{R}(\mathbf{u}_i)$

Newton-Raphson: $\mathbf{K}_{\text{tangent}} \, \Delta \mathbf{u} = -\mathbf{R}(\mathbf{u}_i)$

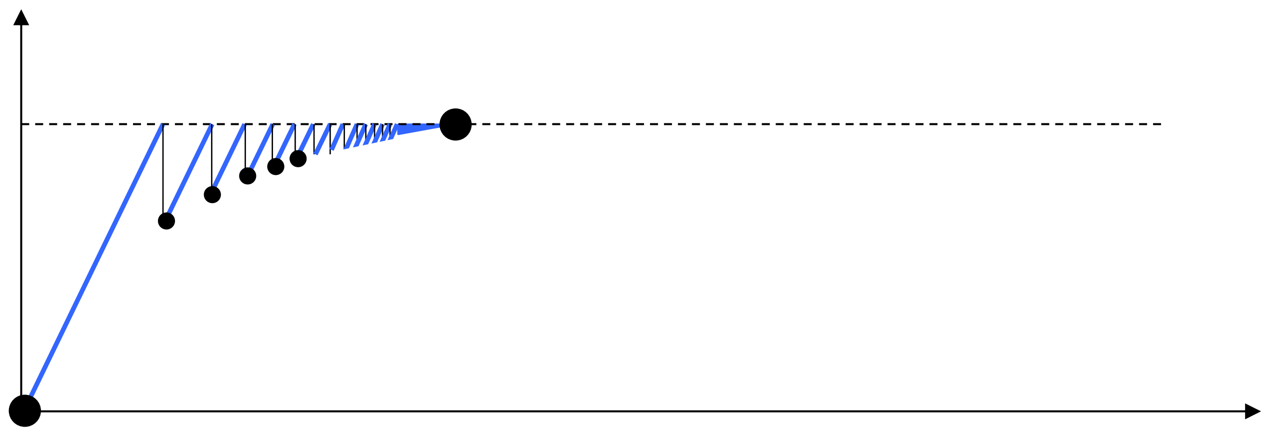
Modified Newton-Raphson: $\mathbf{K}_{\text{initial}} \, \Delta \mathbf{u} = -\mathbf{R}(\mathbf{u}_i)$

The Two Options

Newton-Raphson:

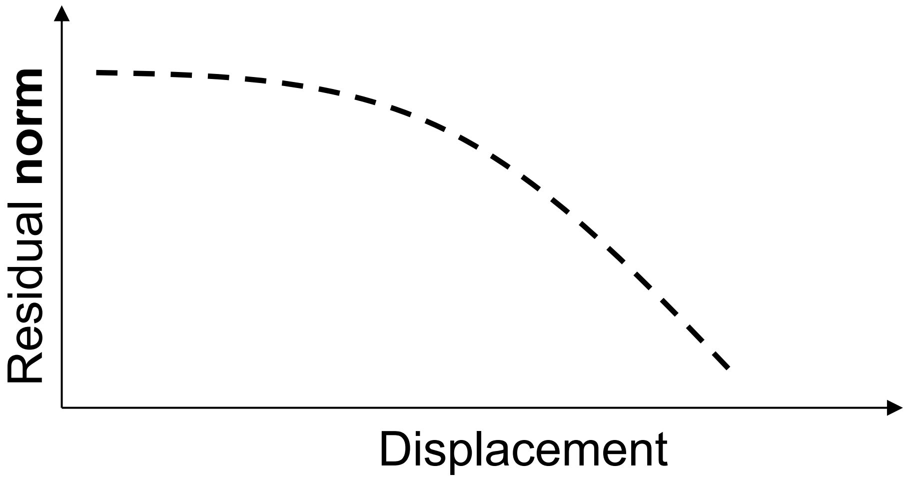


Modified Newton-Raphson:

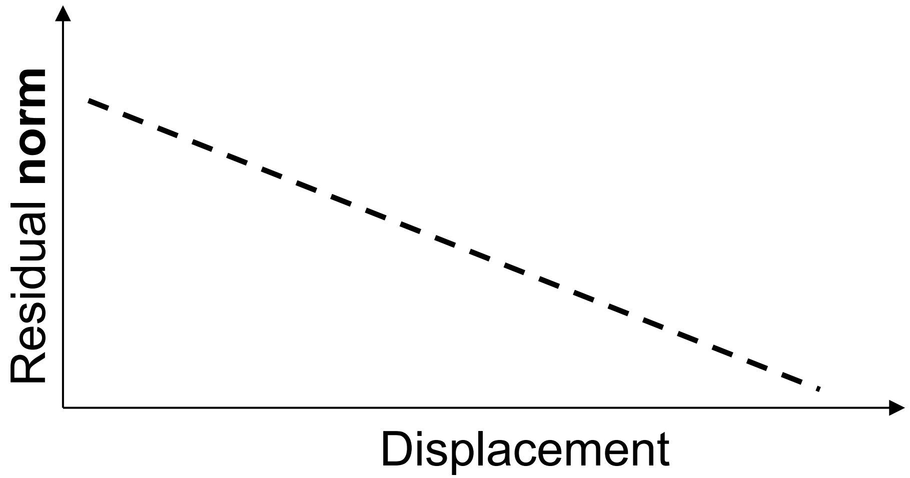


Convergence

Newton-Raphson:



Modified Newton-Raphson:



Python Code

```
while resNorm > tol and i < maxIterations:

    # Check if user wants Modified Newton-Raphson
    if m == stiffnessCalcFrequency:

        m = 0

        # Basic stiffnesses
        Kb1 = (spring1(ub1))[1]
        Kb2 = (spring2(ub2))[1]
        Kb3 = (spring3(ub3))[1]

        # Final stiffness matrix
        Kf = np.transpose(Tbf1 * Kb1).dot(Tbf1) + np.transpose(Tbf2 * Kb2).dot(Tbf2) + np.transpose(Tbf3 * Kb3).dot(Tbf3)

    # Solve for the displacement increment
    duf = np.linalg.solve(Kf, -Rf)

    # New trial displacements
    uf = uf + duf

    # State determination, starting with Basic displacements
    ub1 = np.dot(Tbf1, uf)
    ub2 = np.dot(Tbf2, uf)
    ub3 = np.dot(Tbf3, uf)

    # Basic forces
    Fb1 = (spring1(ub1))[0]
    Fb2 = (spring2(ub2))[0]
    Fb3 = (spring3(ub3))[0]

    # Final force vector
    tildeFf = np.dot(Tbf1.transpose(), Fb1) + np.dot(Tbf2.transpose(), Fb2) + np.dot(Tbf3.transpose(), Fb3)

    # Residual vectdor and its norm
    Rf = tildeFf - Ff
    resNorm = np.linalg.norm(Rf)
```

State Determination

Trial displacements

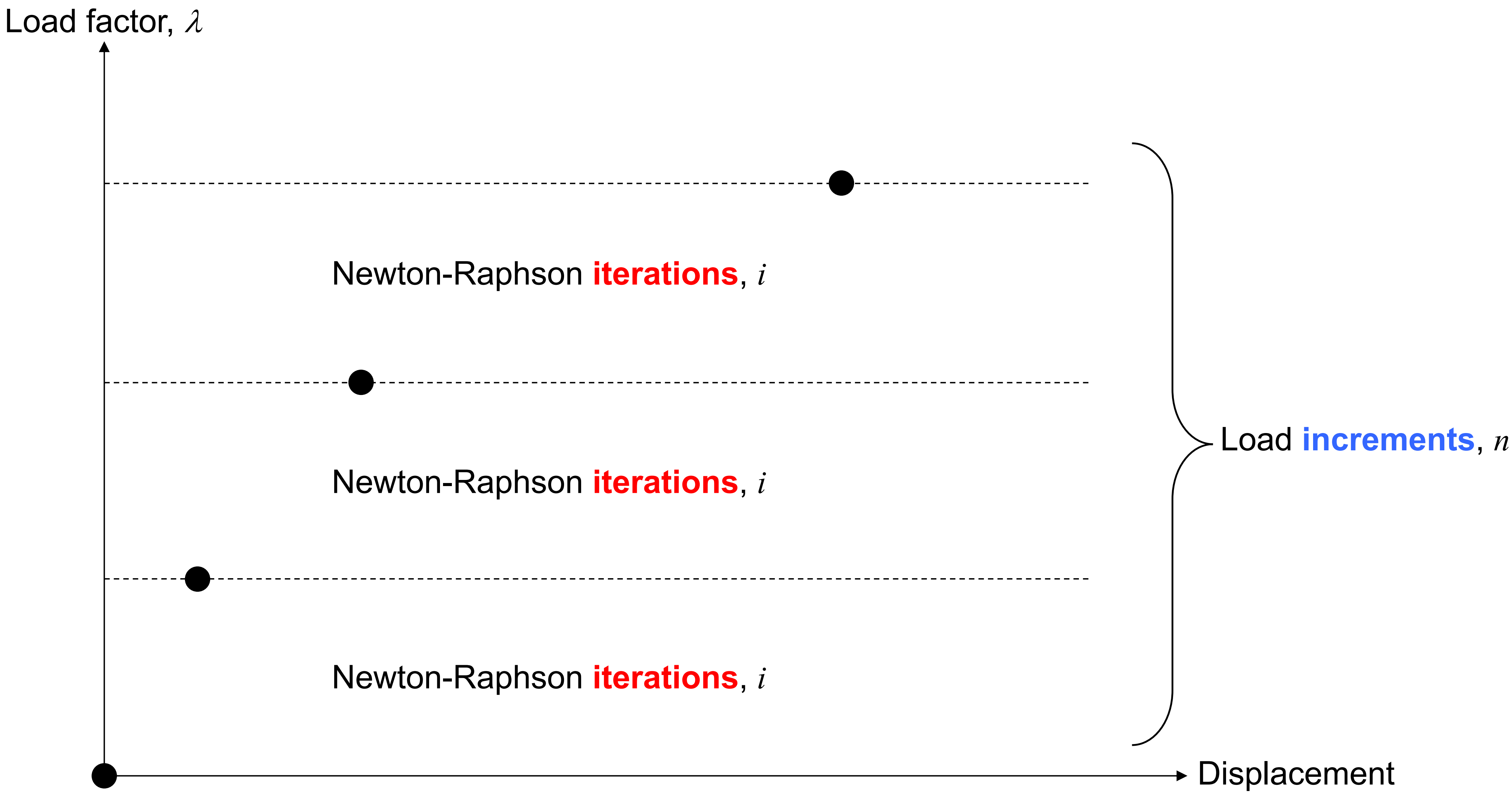
$$\varepsilon = \mathbf{T}_{ms}\mathbf{T}_{sb}\mathbf{T}_{bl}\mathbf{T}_{lg}\mathbf{T}_{ga}\mathbf{T}_{af}\mathbf{u}_f$$

Material model(s) that take total strain, or incremental strain, or both

$$\tilde{\mathbf{F}}_f\left(\mathbf{u}_f\right) = \mathbf{T}_{af}^T \sum \left(\mathbf{T}_{ga}^T \mathbf{T}_{lg}^T \mathbf{T}_{bl}^T \left(\int_0^L \mathbf{T}_{sb}^{\mathrm{T}} \left(\int \mathbf{T}_{ms}^{\mathrm{T}} \cdot \sigma \, dA \right) dx + \bar{\mathbf{F}}_b \right) \right)$$

Check convergence

Iterations vs. Increments

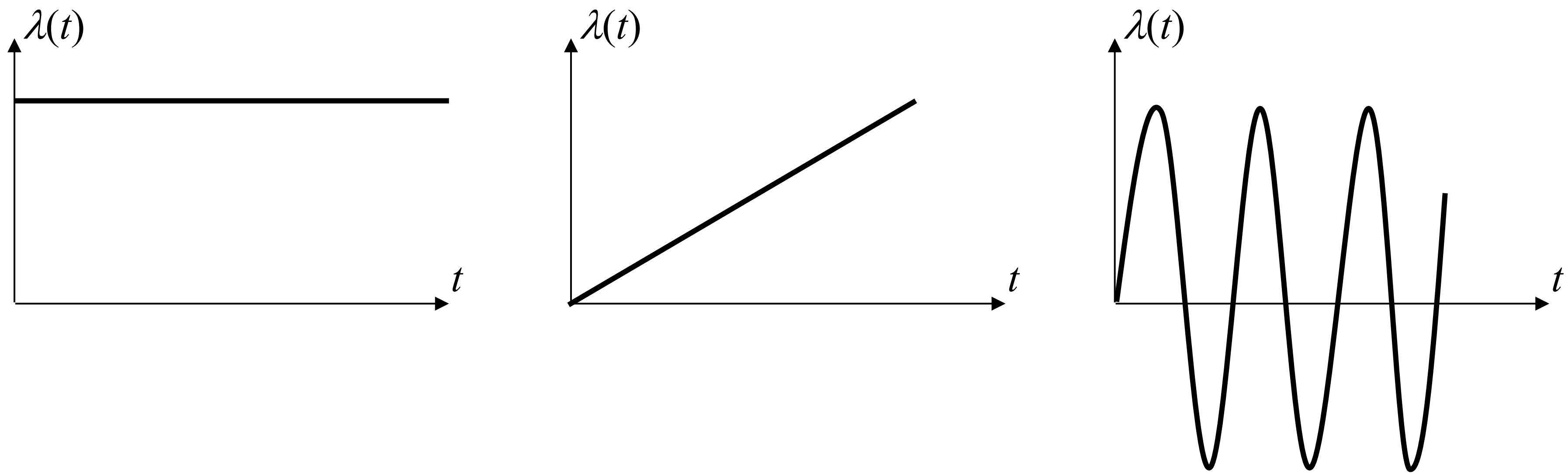


Load Factor, λ

Time series

$$\mathbf{F}_f(t) = \lambda(t) \cdot \mathbf{F}_{ref}$$

Load factor | Pseudo-time | Load pattern



Match Number of Time Steps with Δt

Suppose a 5kN load is to be applied gradually to the structure

Many options!

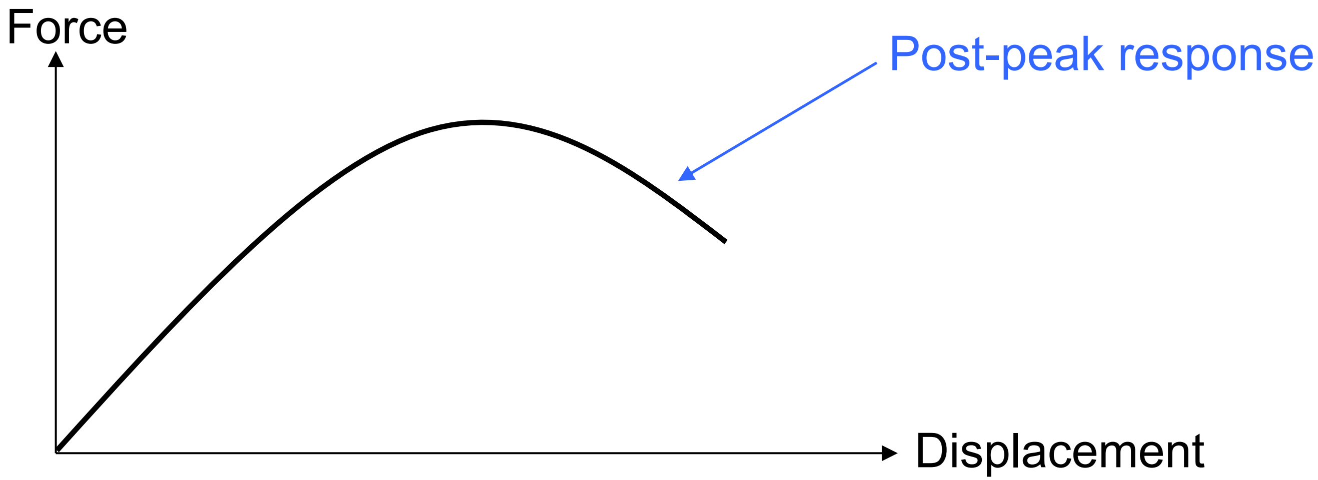
Set $\mathbf{F}_{\text{ref}} = 5\text{kN}$ and $\lambda(t)=t$, reaching the full load at pseudo time $t=1$

Set $\mathbf{F}_{\text{ref}} = 5\text{kN}$ and $\lambda(t)=0.01t$, reaching the full load at time $t=100$

Set $\mathbf{F}_{\text{ref}} = 1\text{kN}$ and $\lambda(t)=t$, reaching the full load at time $t=5$

Set $\mathbf{F}_{\text{ref}} = 1\text{kN}$ and $\lambda(t)=0.1t$, reaching the full load at time $t=50$

Continuation Methods



Load Control during Iterations, Part I

Isolate the change in the load factor in this increment: $\lambda_n = \lambda_{n-1} + \Delta\lambda$

Resulting increment in the load: $\mathbf{F}_n = \mathbf{F}_{n-1} + \Delta\lambda \cdot \mathbf{F}_{ref}$

Resulting expression for the residual: $\mathbf{R}_i = \tilde{\mathbf{F}}_i - \mathbf{F}_n = \tilde{\mathbf{F}}_i - \mathbf{F}_{n-1} - \Delta\lambda \cdot \mathbf{F}_{ref}$

Allow the load factor to vary within the iterations ($n \rightarrow i$): $\mathbf{R}_i = \tilde{\mathbf{F}}_i - \mathbf{F}_{i-1} - \Delta\lambda_i \cdot \mathbf{F}_{ref}$

Steadily accumulating trial displacements: $\mathbf{u}_i = \mathbf{u}_{i-1} + \Delta\mathbf{u}_i$

Linear system of equations for Δu_i: $\mathbf{K}\,\Delta\mathbf{u}_i = -\mathbf{R}_i$

Substitute expression for the residual: $\mathbf{K}\,\Delta\mathbf{u}_i = \mathbf{F}_{i-1} + \Delta\lambda_i \cdot \mathbf{F}_{ref} - \tilde{\mathbf{F}}_i$

Load Control during Iterations, Part II

First term in the split $\Delta\mathbf{u}_i = \Delta\mathbf{u}_{R,i} + \Delta\mathbf{u}_{T,i}$: $\mathbf{K}\,\Delta\mathbf{u}_{R,i} = \mathbf{F}_{i-1} - \tilde{\mathbf{F}}_i$

Namely: $\mathbf{K}\,\Delta\mathbf{u}_{R,i} = \lambda_{i-1} \cdot \mathbf{F}_{ref} - \tilde{\mathbf{F}}_i$

Second term in the split $\Delta\mathbf{u}_i = \Delta\mathbf{u}_{R,i} + \Delta\mathbf{u}_{T,i}$: $\mathbf{K}\,\Delta\mathbf{u}_{T,i} = \Delta\lambda_i \cdot \mathbf{F}_{ref}$

Define reference displacement, constant through iterations: $\mathbf{K}\,\mathbf{u}_T = \mathbf{F}_{ref}$

That means the second term is: $\Delta\mathbf{u}_{T,i} = \Delta\lambda_i \cdot \Delta\mathbf{u}_T$

So the split $\Delta\mathbf{u}_i$ reads: $\Delta\mathbf{u}_i = \Delta\mathbf{u}_{R,i} + \Delta\lambda_i \cdot \mathbf{u}_T$

Displacement Control

Enforce displacement at a control-DOF: One less unknown!

Selection vector: $\mathbf{s} = \{0,0,0,1,0,0,0,0,0\}$

Displacement increment at control-DOF: $\mathbf{s}^{\mathrm{T}}\Delta\mathbf{u}_i = \mathbf{s}^{\mathrm{T}}\Delta\mathbf{u}_{\mathrm{R},i} + \Delta\lambda_i \cdot \mathbf{s}^{\mathrm{T}}\mathbf{u}_{\mathrm{T}}$

First iteration: $\mathbf{s}^{\mathrm{T}}\Delta\mathbf{u}_i = \mathbf{s}^{\mathrm{T}}\Delta\mathbf{u}_{\mathrm{R},i} + \Delta\lambda_i \cdot \mathbf{s}^{\mathrm{T}}\mathbf{u}_{\mathrm{T}} = \Delta u_o$

Solve for $\Delta\lambda_i$, remembering that $\Delta\mathbf{u}_{\mathrm{R},i}$ is zero at first iteration: $\Delta\lambda_1 = \frac{\Delta u_o}{\mathbf{s}^{\mathrm{T}}\mathbf{u}_{\mathrm{T}}}$

Later iterations: $\mathbf{s}^{\mathrm{T}}\Delta\mathbf{u}_i = \mathbf{s}^{\mathrm{T}}\Delta\mathbf{u}_{\mathrm{R},i} + \Delta\lambda_i \cdot \mathbf{s}^{\mathrm{T}}\mathbf{u}_{\mathrm{T}} = 0$

Result: $\Delta\lambda_i = -\frac{\mathbf{s}^{\mathrm{T}}\Delta\mathbf{u}_{\mathrm{R},i}}{\mathbf{s}^{\mathrm{T}}\mathbf{u}_{\mathrm{T}}}$

More lectures:

Terje's Toobox:

terje.civil.ubc.ca