Green-Lagrange Strain

In a document on continuum mechanics, posted near this one, the deformation gradient is employed to define the Green deformation tensor. That, in turn, is employed to define the Lagrange strain tensor, which in index notation reads

$$E_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i} + u_{k,i}u_{k,j}\right) \tag{1}$$

The book by Crisfield and also the book by Hjelmstad, referenced at the end of this document, are excellent references on these matters. Eq. (1) is by far the most common vehicle for introducing geometric nonlinearity in structural analysis. It accommodates moderate but not large displacements and small but not infinitesimally small strains. An important application of Eq. (1) is truss and beam elements, which are "one-dimensional" elements stretching along the *x*-axis. In that case, the relevant axial strain from Eq. (1) is

$$E_{xx} = \frac{1}{2}\left(2 \cdot u_{x,x} + u_{x,x}u_{x,x} + u_{z,x}u_{z,x}\right) = \frac{du}{dx} + \frac{1}{2}\left(\frac{du}{dx}\right)^2 + \frac{1}{2}\left(\frac{dw}{dx}\right)^2 \tag{2}$$

because $u_z=w$. The quantity $(du/dx)^2$ is significantly smaller than the other terms in Eq. (2) and is usually omitted, leading to the following expression for the Green-Lagrange strain:

$$\varepsilon = \frac{du}{dx} + \frac{1}{2}\left(\frac{dw}{dx}\right)^2 \tag{3}$$

The first term in Eq. (3) is the well-known infinitesimal strain from basic structural analysis. It is the second term that is new; it shows how axial strain develops due to rotation, $\theta=dw/dx$, of the element. There are several other ways to derive that expression, addressed in the following sections.

Geometry

A transparent way to derive the Green-Lagrange strain in Eq. (3) is to consider the rotation of the infinitesimally short element in Figure 1. If there is no horizontal displacement of the element ends, only the vertical displacement shown, then the element elongates due to the rotation. That elongation is, from basic trigonometry

$$du \approx dx - dx \cdot \cos(\theta) \tag{4}$$

Next, a series expansion of $\cos(\theta)$ is considered:

$$\cos(\theta) = 1 - \frac{\theta^2}{2} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots \tag{5}$$

Neglecting higher order terms yields

$$\cos(\theta) \approx 1 - \frac{\theta^2}{2} \tag{6}$$

which means that Eq. (4) reads

$$du \approx \frac{1}{2} \cdot \theta^2 \cdot dx \tag{7}$$

Dividing through by *dx* yields

$$\varepsilon = \frac{1}{2}\theta^2 = \frac{1}{2}\left(\frac{dw}{dx}\right)^2 \tag{8}$$

because $\theta = dw/dx$.

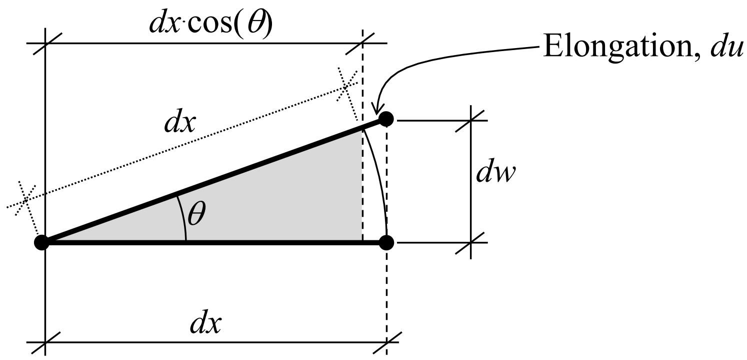

Figure 1: Elongation of rotating truss element.

Engineering Strain

Another way to derive the Green-Lagrange strain is to use the original and deformed element length, visualized in Figure 2. The original element length is L_o and the deformed length is L. For structural engineers, the most intuitive strain expression is the engineering strain

$$\varepsilon_E = \frac{\text{change in length}}{\text{original length}} = \frac{L - L_o}{L_o} \tag{9}$$

That expression can be used to derive the Green-Lagrange strain by letting the finite lengths L and L_o be interchanged with the infinitesimal quantities dx and dw, all visualized in Figure 2. This is done with the understanding that the infinitesimal portion of the strain, $\varepsilon=(L-L_o)/L_o=du/dx$ coming from pure extension, not rotation of the element, is taken care of separately. That said, the theorem of Pythagoras gives the deformed element length caused by rotation, i.e., lateral displacement dw:

$$\varepsilon = \frac{\sqrt{dx^2 + dw^2} - dx}{dx} = \frac{\sqrt{dx^2 + dw^2}}{dx} - 1 = \sqrt{1 + \left(\frac{dw}{dx}\right)^2} - 1 \tag{10}$$

Employing the Taylor series approximation

$$\sqrt{1 + x^2} \approx 1 + \frac{1}{2} \cdot x^2 \tag{11}$$

yields the sought Green-Lagrange strain:

$$\varepsilon = \sqrt{1+\left(\frac{dw}{dx}\right)^2} - 1 \approx \frac{1}{2}\cdot\left(\frac{dw}{dx}\right)^2 \tag{12}$$

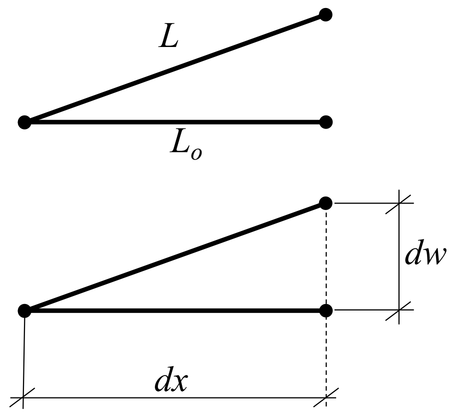

Figure 2: Original and deformed element.

Definition

In an extension of the engineering strain in Eq. (9), the Green-Lagrange strain is actually defined as

$$\varepsilon = \frac{1}{2}\cdot\frac{L^2 - L_o^2}{L_o^2} = \frac{1}{2}\cdot\left(\left(\frac{L}{L_o}\right)^2 - 1\right) \tag{13}$$

That expression reveals that the Green-Lagrange strain essentially adds a second-order term to the engineering strain:

$$\varepsilon = \frac{1}{2}\cdot\left(\left(\frac{L}{L_o}\right)^2 - 1\right) = \frac{1}{2}\cdot\left(\left(\frac{L-L_o}{L_o}+\frac{L_o}{L_o}\right)^2 - 1\right) = \frac{1}{2}\cdot((\varepsilon_E+1)^2-1) = \varepsilon_E + \frac{1}{2}\cdot\varepsilon_E^2 \tag{14}$$

That is to say that the Green-Lagrange strain is defined as

$$\varepsilon = \frac{L-L_0}{L_0} + \frac{1}{2}\left(\frac{L-L_0}{L_0}\right)^2 \tag{15}$$

Returning to Eq. (13), we can show that the portion of the axial strain coming from rotation of the element is indeed captured by that formula, using the theorem of Pythagoras in order to determine the element length:

$$\varepsilon = \frac{1}{2}\left(\left(\frac{\sqrt{dx^2+dw^2}}{dx}\right)^2 - 1\right) = \frac{1}{2}\left(\frac{dx^2+dw^2}{dx^2} - 1\right) = \frac{1}{2}\left(1+\left(\frac{dw}{dx}\right)^2 - 1\right) = \frac{1}{2}\left(\frac{dw}{dx}\right)^2 \tag{16}$$

Final Strain Expressions for Truss and Beam Elements

For reference, the infinitesimal strain for a truss element is

$$\varepsilon = \frac{du}{dx} \tag{17}$$

and the infinitesimal strain for an Euler-Bernoulli beam element is

$$\varepsilon = z \cdot \frac{d^2 w}{dx^2} \tag{18}$$

Introducing the new strain contribution, from rotation of the element, derived above, gives

$$\varepsilon = \frac{du}{dx} + \frac{1}{2} \cdot \left(\frac{dw}{dx} \right)^2 \tag{19}$$

for the truss element and

$$\varepsilon = z \cdot \frac{d^2 w}{dx^2} + \frac{1}{2} \cdot \left(\frac{dw}{dx} \right)^2 \tag{20}$$

for the beam element. Those enhanced strain expressions can be used in hand calculations with energy methods and Ritz approximations, or in the principle of virtual displacements in order to derive finite elements with geometric nonlinearity.

References

Hjelmstad, K. D. (1997). “Fundamentals of Structural Mechanics.” Prentice Hall.

Crisfield (1991). “Nonlinear Finite Element Analysis of Solids and Structures.” Volume 1. First Edition. Wiley.

de Borst, **Crisfield**, Remmers, Verhoosel (2012). “Nonlinear Finite Element Analysis of Solids and Structures.” Second Edition. Wiley.