Geometry and Trigonometry

Dot Product: Length and Area

Consider two vectors **a**, **b** $\in \mathbb{R}^3$. The dot product is

$$\mathbf{a} \bullet \mathbf{b} = \mathbf{a}^T \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3 = \|\mathbf{a}\| \cdot \|\mathbf{b}\| \cdot \cos(\theta) \quad (1)$$

where θ is the angle between the vectors and $||\mathbf{a}||$ is the vector norm, i.e., the length of the vector, i.e., $\|\mathbf{a}\| = \sqrt{x_1^2 + x_2^2 + x_3^2}$. If one of the vectors has unit length then the dot product reveals the length of the other vector in that direction. The dot product between two orthogonal (perpendicular) vectors is zero. The dot product is related to the area of the triangle that is spanned by the two vectors:

$$A = \frac{1}{2} \cdot \|\mathbf{a}\| \cdot \|\mathbf{b}\| \cdot \sin(\theta) \quad (2)$$

Cross Product: Area

The cross product, sometimes called the vector product, yields a vector:

$$\mathbf{a} \times \mathbf{b} = \begin{Bmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{Bmatrix} \quad (3)$$

The norm of the cross-product vector is equal to twice the area of the triangle that is spanned by the two vectors:

$$A = \frac{1}{2} \cdot \|\mathbf{a} \times \mathbf{b}\| \quad (4)$$

Consequently, the cross product between two parallel vectors yields a vector of zeros. Use the right hand to identify the direction of the cross product result relative to **a** and **b**:

- Point the index finger straight forward and let it represent the direction of **a**
- Bend the middle finger 90° to the index finger and let it represent the direction of **b**
- The thumb is perpendicular to **a** and **b** and shows the direction of the cross product vector

Observe that the cross product between two vectors that lie in the 1-2 plane is non-zero only in the 3-direction, with value $a_1 b_2 - a_2 b_1$. Hence, the area of a triangle that lies in the 1-2 plane is $\frac{1}{2}|a_1 b_2 - a_2 b_1|$, where **a** and **b** define two of the sides.

Angle

Angles are measured in radians (from 0 to 2π) or degrees (from 0° to 360°). An angle in radians is the ratio of the length of a circle segment to the radius of that circle:

$$\theta = \frac{\text{Arc length}}{\text{Radius}} \tag{5}$$

Note that in any triangle, the sum of the three corner angles is π=180°. In terms of a dot product, the following relationship holds:

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| \cdot |\mathbf{b}| \cdot \cos(\theta) \tag{6}$$

In terms of a cross product, the following relationship holds:

$$\mathbf{a} \times \mathbf{b} = |\mathbf{a}| \cdot |\mathbf{b}| \cdot \sin(\theta) \tag{7}$$

Trigonometry: Sine, Cosine, Tangent

Trigonometry is a field within geometry devoted to the study of triangles. Consider a right-angled triangle, i.e., a triangle with one angle equal to 90°. Let θ denote one of the other two angles and let the word *Opposite* identify the length of the side of the triangle that is opposite of that angle. Name the length of the longest side *Hypotenuse*. Finally, let *Adjacent* identify the length of the remaining side. The trigonometric functions are defined as

$$\cos(\theta) = \frac{Adjacent}{Hypotenuse} \tag{8}$$

$$\sin(\theta) = \frac{Opposite}{Hypotenuse} \tag{9}$$

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} = \frac{Opposite}{Adjacent} \tag{10}$$

Also defined is the cotangent:

$$\cot(\theta) = \frac{Adjacent}{Opposite} = \frac{1}{\tan(\theta)} \tag{11}$$

the secant:

$$\sec(\theta) = \frac{Hypotenuse}{Adjacent} = \frac{1}{cos(\theta)} \tag{12}$$

and the cosecant

$$\csc(\theta) = \frac{Hypotenuse}{Opposite} = \frac{1}{\sin(\theta)} \tag{13}$$

Pythagoras

Pythagoras' rule is valid for right-angled triangles and can be proven by rearranging a collection of such triangles in different ways, ultimately finding that

$$a^2 + b^2 = c^2 \tag{14}$$

where a and b are the lengths of the short sides while c is the length of the longest side.

Law of Cosines

The law of cosines is an extension of the rule of Pythagoras to triangles that are not right-angled and states that

$$a^2 + b^2 - 2 \cdot a \cdot b \cdot \cos(\gamma) = c^2 \tag{15}$$

where the "correction factor" $2ab\cos(\gamma)$ compared with the rule of Pythagoras contains the angle γ measured at the corner opposite the length c.

Spherical Law of Cosines

The spherical law of cosines is modified as follows when the triangle is laid on the surface of a sphere with unit radius:

$$\cos(a) \cdot \cos(b) + \sin(a) \cdot \sin(b) \cdot \cos(\gamma) = \cos(c) \tag{16}$$

Because the sphere has unit radius the lengths a, b, and c are angles measured at the centre of the sphere. For non-unit spheres, a, b, and c should be thought of as those angles. When the triangle is right-angled then the angle γ is 90 degrees and hence $\cos(\gamma)=0$ so that the spherical version of the rule of Pythagoras is obtained:

$$\cos(a) \cdot \cos(b) = \cos(c) \tag{17}$$

Haversine: Great Circle Distance

Due to historical developments in navigation, and also due to potential round off in calculations with the spherical law of cosines, the haversine formula is popular for calculating the "great-circle distance" between two points on a sphere given longitudes and latitudes. Using the notation from above the law of haversines states the spherical law of cosines:

$$\text{hav}(a-b) + \sin(a) \cdot \sin(b) \cdot \text{hav}(\gamma) = \text{hav}(c) \tag{18}$$

where the haversine function is

$$\text{hav}(\theta) = \sin^2\left(\frac{\theta}{2}\right) = \frac{1-\cos(\theta)}{2} \tag{19}$$

The spherical law of cosines is recovered by also recalling that

$$\cos(a-b) = \cos(a) \cdot \cos(b) + \sin(a) \cdot \sin(b) \tag{20}$$

The law of haversines is specialized to the haversine formula for calculation of distances based on longitudes (λ_1, λ_2) and latitudes (φ_1, φ_2) by moving to the North Pole the point where γ is measured. In fact, γ is then the longitude separation between the two points whose distance, c, is sought, while a and b are the latitude from the North Pole down to those points. Because latitudes are measured from equator we have $a=\pi/2-\varphi_1$ so that $\sin(a)$ is replaced with $\cos(\varphi_1)$ in the application of the law of haversines. When further multiplying the angle c by the Earth's radius r to get the sought distance d the haversine formula reads

$$\text{hav}\left(\frac{d}{r}\right) = \text{hav}(\varphi_2 - \varphi_1) + \cos(\varphi_1)\cdot\cos(\varphi_2)\cdot\text{hav}(\lambda_2 - \lambda_1) \tag{21}$$

Using only the standard trigonometric functions the haversine formula can be solved in several ways, for example to calculate the distance between two points on Earth as follows:

$$d = r\cdot\text{acos}\big(\sin(\varphi_1)\cdot\sin(\varphi_2) + \cos(\varphi_1)\cdot\cos(\varphi_2)\cdot\cos(\lambda_2 - \lambda_1)\big) \tag{22}$$

typically with r=6,371km.

Points, Lines, Planes

A point, say **s**, is a vector that contains the coordinates of the point. In the following, the vector **x** denotes a generic point with coordinates x_1, x_2, x_3 or x, y, z. A direction, say **d**, is a vector that contains the relative length of the vector in each coordinate direction. A line is defined in terms of a point, a direction, and a scalar variable, here named t:

$$\mathbf{l} = \mathbf{s} + \mathbf{d}\cdot t \tag{23}$$

A plane, i.e., a linear surface, is defined by

$$h(\mathbf{x}) = a\cdot x_1 + b\cdot x_2 + c\cdot x_3 + d = 0 \tag{24}$$

The normal to the plane is $\mathbf{n}=\nabla h=\{a\ b\ c\}^T$. Hence, one way to construct a plane is to identify a normal vector, which directly gives a, b, and c, and use some reference point on the surface to determine d. Conversely, to construct a plane that intersects three points the cross product between the two identifiable vectors is computed, which gives the normal to the plane. Subsequently, any of the three points determine d. The potential point of intersection between two lines is obtained by equating two equations of the form in Eq. (23). The distance Δ from a point **p** to a line **l** can be found in two ways. First, denote by **c** the point on the line that is closest to **p**. The vector **p**-**c** must be perpendicular to the line **l**. Hence, the unknown line parameter, denoted t in Eq. (23) is solved from

$$\mathbf{l}^T(\mathbf{p} - \mathbf{c}) = 0 \tag{25}$$

The alternative approach combines Eqs. (2) and (4). Consider the triangle between the points **s**, **p**, and **c**, and let θ denote the angle at $\mathbf{x}_0$. It follows that

$$\sin(\theta) = \frac{\Delta}{\|\mathbf{p} - \mathbf{s}\|} \tag{26}$$

The area of the triangle by Eqs. (2) and (4) is

$$A = \frac{1}{2}\cdot\|(\mathbf{p} - \mathbf{s})\times(\mathbf{c} - \mathbf{s})\| = \frac{1}{2}\cdot\|\mathbf{p} - \mathbf{s}\|\cdot\|\mathbf{c} - \mathbf{s}\|\cdot\sin(\theta) \tag{27}$$

Combination of Eq. (26) and (27) provides the sought distance:

$$\Delta = \frac{\|(\mathbf{p} - \mathbf{s})\times(\mathbf{c} - \mathbf{s})\|}{\|\mathbf{c} - \mathbf{s}\|} \tag{28}$$

The distance Δ from a point **p** to a plane $h(\mathbf{x})$=0 is

$$\Delta = \left| \frac{h(\mathbf{p})}{\|\nabla h\|} \right| = \left| \frac{a \cdot p_1 + b \cdot p_2 + c \cdot p_3 + d}{\sqrt{a^2 + b^2 + c^2}} \right| \tag{29}$$

This formula is derived as follows. Consider a point **q** that lies in the plane $h(\mathbf{x})=0$, i.e., $a \cdot q_1 + b \cdot q_2 + c \cdot q_3 + d = 0$. The normal vector ∇h and the vector (**p-q**), the latter having length Δ, are parallel, i.e., $\cos(\theta)=\pm 1$ in the dot product between them. In fact, the dot product between these two vectors is evaluated in to ways suggested by Eq. (1):

$$\nabla h \bullet (\mathbf{p} - \mathbf{q}) = \|\nabla h\| \cdot \Delta \cdot (\pm 1) \tag{30}$$

$$\begin{aligned} (\nabla h)^T (\mathbf{p} - \mathbf{q}) &= a \cdot (p_1 - q_1) + b \cdot (p_2 - q_2) + c \cdot (p_3 - q_3) \\ &= a \cdot p_1 + b \cdot p_2 + c \cdot p_3 - (a \cdot q_1 + b \cdot q_2 + c \cdot q_3) \\ &= a \cdot p_1 + b \cdot p_2 + c \cdot p_3 - d \end{aligned} \tag{31}$$

Equating Eqs. (30) and (31) yields Eq. (29).

Rotation of a Coordinate System

The rotation of a Cartesian coordinate system appears in several problems addressed in the notes on this website:

- Rotation of the 2D *y-z* axis system of a cross-section if the original axes are not the principal axes
- Transformation of element degrees of freedom from the "local" to the "global" configuration; both 2D and 3D coordinate systems are within scope
- Rotation of the multi-dimensional axis system in the standard normal space of random variable in the second-order reliability method (SORM)

It is noted in passing that the "rotation matrix," **R**, employed in the rotation of coordinate systems is the transpose and at the same time inverse of the "transformation matrix," **T**, employed in the contragredient transformation in matrix structural analysis when only displacement degrees of freedom are present:

$$\mathbf{x}_{global} = \mathbf{R}\mathbf{x}_{local} \quad \text{and} \quad \mathbf{u}_{local} = \mathbf{T}\mathbf{u}_{global} \quad \text{with} \quad \mathbf{T} = \mathbf{R}^T \tag{32}$$

where the global coordinate system will be denoted $\bar{\mathbf{x}}$ in the following, where the objective is to rotate coordinates from the **x**-system to the $\bar{\mathbf{x}}$-system. Figure 1 is included to show that the sought rotation can be derived manually, at least in the 2D case. As shown in Figure 1 the result is:

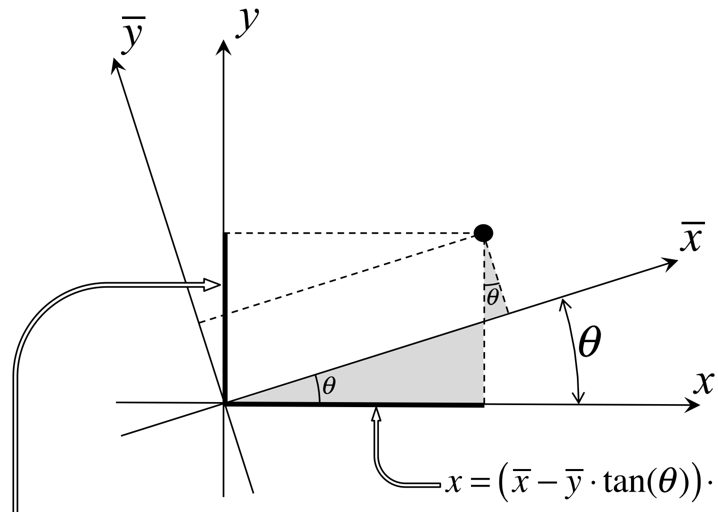
$$\begin{aligned} x &= \bar{x} \cdot \cos(\theta) - \bar{y} \cdot \sin(\theta) \\ y &= \bar{x} \cdot \sin(\theta) + \bar{y} \cdot \cos(\theta) \end{aligned} \Rightarrow \begin{Bmatrix} x \\ y \end{Bmatrix} = \underbrace{\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}}_{=\mathbf{T} = \mathbf{R}^T = \mathbf{R}^{-1}} \begin{Bmatrix} \bar{x} \\ \bar{y} \end{Bmatrix} \tag{33}$$

In other words, for 2D rotations, the coordinates of a rotated system is obtained from the equation

$$\bar{\mathbf{x}} = \mathbf{R}\mathbf{x} \tag{34}$$

where the rotation matrix is

$$\mathbf{R} = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} \tag{35}$$



$$x = \left(\bar{x} - \bar{y}\cdot\tan(\theta)\right)\cdot\cos(\theta)$$
$$= \bar{x}\cdot\cos(\theta) - \bar{y}\cdot\sin(\theta)$$

$$y = x\cdot\tan(\theta) + \frac{\bar{y}}{\cos(\theta)}$$
$$= \left(\bar{x}\cdot\cos(\theta) - \bar{y}\cdot\sin(\theta)\right)\cdot\tan(\theta) + \frac{\bar{y}}{\cos(\theta)}$$
$$= \bar{x}\cdot\sin(\theta) - \bar{y}\cdot\left(\sin(\theta)\cdot\tan(\theta) - \frac{1}{\cos(\theta)}\right)$$
$$= \bar{x}\cdot\sin(\theta) - \bar{y}\cdot\frac{1}{\cos(\theta)}\left(\sin^2(\theta) - 1\right)$$
$$= \bar{x}\cdot\sin(\theta) + \bar{y}\cdot\cos(\theta)$$

Figure 1: Manuel rotation for 2D coordinate system.

To understand the 3D case, start by considering the orthonormal basis vectors $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ of a coordinate system that is here nicknamed the "local" system. Furthermore, let $\mathbf{n}_1$, $\mathbf{n}_2$, and $\mathbf{n}_3$ be the orthonormal basis vectors of another coordinate system, here called the "global" system. In the two coordinate systems a vector **v** is written:

$$\mathbf{v} = x\cdot\mathbf{e}_1 + y\cdot\mathbf{e}_2 + z\cdot\mathbf{e}_3 = \bar{x}\cdot\mathbf{n}_1 + \bar{y}\cdot\mathbf{n}_2 + \bar{z}\cdot\mathbf{n}_3 \tag{36}$$

where x, y, z, $\bar{x}$, $\bar{y}$, and $\bar{z}$ are constants. If the vector **v** starts at the origin then these constants are the coordinates of the other end of the vector. That is, **x**={x, y, z} are the coordinates in the local system and $\bar{\mathbf{x}} = \{\bar{x}, \bar{y}, \bar{z}\}$ are the coordinates of the same point in the global system. The rotation matrix transforms coordinates from the local to the global system:

$$\bar{\mathbf{x}} = \mathbf{R}\mathbf{x} \tag{37}$$

The rotation matrix, **R**, consists of direction cosines:

$$\begin{Bmatrix} \bar{x} \\ \bar{y} \\ \bar{z} \end{Bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix} \begin{Bmatrix} x \\ y \\ z \end{Bmatrix} \tag{38}$$

where c_{ij} is the direction cosine between $\mathbf{n}_i$ and $\mathbf{e}_k$, i.e., the dot product between those two basis vectors. One approach for establishing **R** in the context of a uniaxial structural element, such as a truss or beam element, is to start with the direction cosines of the element, namely

$$c_x = \frac{\Delta x}{L}, \quad c_y = \frac{\Delta y}{L}, \quad c_z = \frac{\Delta z}{L} \tag{39}$$

where Δx, Δy, and Δz are the projections of the element along the x, y, and z axes, respectively. The vector consisting of these direction cosines forms the first column in **R**, whose columns consists of three orthogonal vectors, **x**, **y**, **z**:

$$\mathbf{R} = \begin{bmatrix} \mathbf{x} & \mathbf{y} & \mathbf{z} \end{bmatrix} \tag{40}$$

where **x** is a vector that contains the direction cosines of Eq. (39). The vectors **y** and **z** are expressed as the cross products

$$\mathbf{y} = \mathbf{v} \times \mathbf{x} \tag{41}$$

$$\mathbf{z} = \mathbf{x} \times \mathbf{y} \tag{42}$$

where **v** is a user-defined vector that orients the local *z*-axis of the element in the global coordinate system. As an example, consider a frame element that, as usual, has the local *x*-axis along the element length. Suppose the end nodes of this element are positioned in the global coordinate system so that the element is oriented parallel to the global *z*-axis. Practically, this means that the element is a vertical column if the global *z*-axis defines the upward direction. To orient the local *z*-axis of the element along the global *x*-direction the vector **v** is given as $\{1\ 0\ 0\}$. Conversely, to orient the local *z*-axis along the global y-direction the vector **v** is given as $\{0\ 1\ 0\}$. Given the element orientation it is impossible to orient the local *z*-axis along the global *z*-direction. When providing the vector **v** it is sufficient to give a direction in the global system that lies in the *x*-*z*-plane of the local element. That is, the vectors $\{1\ 0\ 0\}$ and $\{1\ 0\ 0\}$ would work fine even if the element is not oriented completely vertical in the global system. Importantly, however, all the vectors **v**, **y**, and **z** must have unit length; **x** is already normalized by its definition in terms of the direction cosines. As a practical interpretation of **R**, note that the vectors **x**, **y**, and **z** represent orthogonal vectors in the global coordinate system. For truss and frame elements the vector **x** is aligned with the longitudinal element axis, which is given by the direction cosines of the element. The **y** and **z** vectors rotate the element around the longitudinal axis by means of the auxiliary vector **v**. For example, to rotate a beam element about its local x-axis by an angle θ, two rotation matrices can be applied. First, a rotation matrix with the following vector **v** is applied to rotate the cross-section around the *x*-axis by the angle θ:

$$v = \begin{Bmatrix} 0 \\ -\sin(\theta) \\ \cos(\theta) \end{Bmatrix} \tag{43}$$

Next, the rotation matrix in Eq. (40) is applied, with a "standard" **v** that is always pointing in the global *z*-direction, unless the element is parallel to that axis, in which case **v** can be taken to point, say, in the global negative *x*-direction.