Geometric Nonlinearity

The material presented in this document is brief, covering mostly the grand scheme of things. For details and perhaps more clarity, check the following material posted on this website:

- Linearized theory
  - **Document**: Frame Elements with P-delta
  - **Python code**: G2 Element 6
- Nonlinear theory
  - **Document**: Truss Element with Geometric Nonlinearity
  - **Python code**: G2 Element 3

When loads are applied to a linear elastic structure, it deforms. That deformation may in turn cause additional forces and deformations in the structure. Those added effects, which are neglected in ordinary structural analysis, can be introduced in two different ways.

Revised Equilibrium

One approach is to consider equilibrium in a displaced configuration. That is done in the documents Beams with Axial Force and Frame Elements with P-Delta posted on this website. The result is a modified element stiffness, featuring a geometric stiffness contribution in addition to the ordinary elastic stiffness. The classical example is a "leaning column" with an axial force, *P*, which causes an overturning moment because of lateral displacement, Δ. That overturning moment is P·Δ and the inclusion of that effect reduces the lateral stiffness, when *P* is a compressive force. In general, the stiffness modification reads

$$[\mathbf{K} - P \cdot \mathbf{K}^{\mathrm{G}}]\mathbf{u} = \mathbf{F} \tag{1}$$

where **K**=elastic stiffness matrix and $\mathbf{K}^{\mathrm{G}}$=geometric stiffness matrix. Eq. (1) shows that the total stiffness of the structure diminishes as the compressive axial force, *P*, increases. Eq. (1) also shows that the inclusion of P-Delta effects is still a linear problem, if *P* is constant. In other words, this is *not geometric nonlinearity* as explained in the next section. Rather, the inclusion of P-delta effects via equilibrium in the displaced configuration is a linearized theory, resulting in changed stiffness and the possibility of solving the eigenvalue problem that Eq. (1) represents when **F**=**0**. The eigenvalues are the buckling loads and the eigenvectors are the corresponding shapes of the structure for each buckling load. Three levels of granularity are possible when introducing P-Delta effects:

- **Big P-Delta:** This is the leaning column effect associated with the geometric stiffness *P*/*L*. A simple stick model of a column with axial force reveals this effect. It is the largest P-Delta effect and in engineering practice it is the most important effect to include.
- **Small P-delta:** This is the effect of the bending of the column and manifests in the components of the geometric stiffness matrix beyond *P*/*L*; see the document on Frame Elements with P-Delta.

- **Tiny P-delta:** This is the effect obtained by going beyond the document on Frame Elements with P-Delta, i.e., beyond the approximate polynomial shape functions, to the utilization of the exact stability functions. However, if this is done, equilibrium can no longer be written in the form of Eq. (1), removing the possibility of solving a classical eigenvalue problem for the buckling loads.

Revised Kinematic Compatibility

Another approach to introduce effects of deformations is to revise the kinematic compatibility considerations. That means modifying the strain expressions. Interestingly, there is no unique strain expressions when we leave the realm of infinitesimal strains. Rather a choice must be made, and that choice may depend on the behaviour of the material and the extent of the deformations. In order to introduce moderate but not large displacements, while assuming small but not infinitesimally small strains, the Green-Lagrange strain is by far the most common choice. It is described in a separate document on this website and epitomized by the expression

$$\varepsilon = \frac{1}{2}\theta^2 = \frac{1}{2}\cdot\left(\frac{dw}{dx}\right)^2 \tag{2}$$

which shows axial strain caused by rotation of an element. This is the fundamental vehicle for introducing *geometric nonlinearity*. Finite elements with that feature are developed by substituting the extended strain expression into the principle of virtual displacements.

Coordinate System

When new strain expressions are employed to include geometric nonlinearity, questions quickly emerge about the coordinate system in which those strains are expressed. Structural engineers are most familiar with the Lagrangian approach. That is a solid-mechanics approach, in which the coordinate system remains fixed while the material moves within the coordinate system. In contrast, hydrotechnical engineers may be more familiar with the Eulerian approach, in which the coordinate system follows the material particle, e.g., fluid volume, as it moves. When developing finite elements with geometric nonlinearity, using the Green-Lagrange strain from the previous section, three options are available:

- **Total Lagrangian:** The original coordinate system remains fixed.
- **Updated Lagrangian:** The developments are made with reference to a coordinate system aligned with the previously converged equilibrium state of the element.
- **Corotational:** The coordinate system moves with the element.

Conservative vs. Unconservative Loads

The loads applied to the structure are assumed to retain their loading direction, regardless of the amount of deformation. That is called conservative loads. So, far none of the documents on this website deals with anything else. For example, if the vertical load on a vertical column remains vertical when the column deforms, then it is a conservative load. The more advanced case is unconservative, i.e., “follower loads” that change direction when the structure deforms. Those are harder to analyze, and renders the variational form of the boundary value problem unavailable.

References

Crisfield (1991). “Nonlinear Finite Element Analysis of Solids and Structures.” Volume 1. First Edition. Wiley.

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