Distributed Plasticity Elements

Many of the notes posted on this website arrange equilibrium, compatibility, and material law as shown in Figure 1. The left-hand side contains equilibrium and the right-hand side contains compatibility equations. Figure 1 also adopts the following notation from other documents: **u**=degrees of freedom (DOFs); **F**=force/load vector; **K**=stiffness matrix; and **T**=transformation matrix. Also, following notation established in other documents, the nonlinear resisting (restoring) forces are identified by a tilde symbol: $\tilde{\mathbf{F}}$. In the document on the computational stiffness method the element "configurations" Basic, Local, Global are explained; what is new in Figure 1 is the Section and Material levels. Those levels are vital in the nonlinear analysis with frame elements that is described in this document.

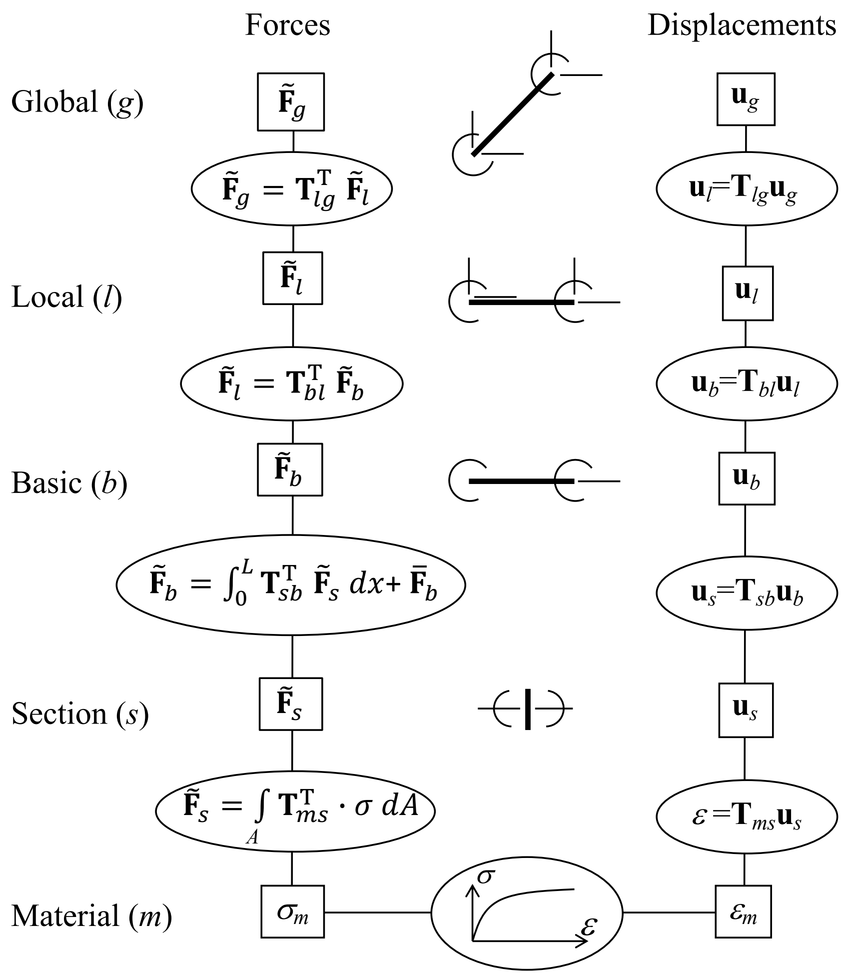

Figure 1: Notation.

State Determination

The state determination in nonlinear analysis requires the determination of the resisting forces, and often the tangent stiffness, associated with the trial displacements at the structural level, $\mathbf{u}_f$. The global element displacements, $\mathbf{u}_g$, are determined without transformation matrices, e.g., using the ID array described in the document on the computational stiffness method. Next, regardless of whether "displacement-based" or "force-based" elements are employed (described below) the basic deformations are

$$\mathbf{u}_b = \mathbf{T}_{bl}\mathbf{T}_{lg}\mathbf{u}_g = \mathbf{T}_{bg}\mathbf{u}_g = \begin{bmatrix} -\frac{\Delta x}{L} & -\frac{\Delta y}{L} & 0 & \frac{\Delta x}{L} & \frac{\Delta y}{L} & 0 \\ \frac{\Delta y}{L^2} & -\frac{\Delta x}{L^2} & 1 & -\frac{\Delta y}{L^2} & \frac{\Delta x}{L^2} & 0 \\ \frac{\Delta y}{L^2} & -\frac{\Delta x}{L^2} & 0 & -\frac{\Delta y}{L^2} & \frac{\Delta x}{L^2} & 1 \end{bmatrix}\mathbf{u}_g \tag{1}$$

where the transformation matrix is taken from the document on the computational stiffness method and given for a 2D element with Δx and Δy being the x- and y-distances between the end points. Having established the link between the basic and global element configurations, attention turns to the link between the cross-section and the basic configuration, i.e., moving further down Figure 1. It is the transformation between the basic and section degrees of freedom that differ between the displacement-based and force-based elements.

Displacement Interpolation

The "displacement-based element" of nonlinear analysis is based on the typical use of shape functions in finite element analysis, i.e., to discretize the displacement field. For example, for a beam element:

$$w(x) = \mathbf{N}\mathbf{u}_b \tag{2}$$

where w=transversal displacement, x=longitudinal axis, $\mathbf{N}$=shape functions, and $\mathbf{u}_b$=displacements and rotations along the degrees of freedom. For frame elements the shape functions in $\mathbf{N}$ are usually third-order polynomials. For the degrees of freedom shown in Figure 2 those functions are

$$N_1(x) = \frac{x}{L} \tag{3}$$

$$N_2(x) = -\frac{1}{L^2}x^3 + \frac{2}{L}x^2 - x \tag{4}$$

$$N_3(x) = -\frac{1}{L^2}x^3 + \frac{1}{L}x^2 \tag{5}$$

Figure 2: Basic degrees of freedom.

With those shape functions the section deformations are

$$\mathbf{u}_s = \begin{Bmatrix} \varepsilon \\ \kappa \end{Bmatrix} = \begin{Bmatrix} u' \\ w'' \end{Bmatrix} = \begin{bmatrix} N_1' & 0 & 0 \\ 0 & N_2'' & N_3'' \end{bmatrix} \mathbf{u}_b = \mathbf{T}_{sb}\mathbf{u}_b \tag{6}$$

where ε=axial strain and κ=curvature corresponding to the section forces are $\mathbf{F}_s=\{N, M\}$. Evaluating Eq. (6) yields

$$\mathbf{T}_{sb} = \begin{bmatrix} \frac{1}{L} & 0 & 0 \\ 0 & -\left(\frac{6\cdot x}{L^2} - \frac{4}{L}\right) & \left(\frac{2}{L} - \frac{6\cdot x}{L^2}\right) \end{bmatrix} \tag{7}$$

The behaviour of the material can be modelled as a cross-section model, in which case the state determination now has done what is needed for the material model to determine the section forces. However, in the following, it is assumed that a "fibre-discretized" cross-section model is employed. That means the cross-section consists of a collection of uniaxial material models. Once the section deformations are determined, the strain in each fibre is obtained with the transformation:

$$\varepsilon_m = \mathbf{T}_{ms}\mathbf{u}_s = \begin{Bmatrix} 1 & -z \end{Bmatrix} \begin{Bmatrix} \varepsilon \\ \kappa \end{Bmatrix} \tag{8}$$

where z is the distance from the neutral axis to the fibre in which the stress is calculated. This is shown in the Python code for G2 posted on this website, implementing a fibre-discretized cross-section. The state determination now proceeds to determine the Basic element forces. Once the material stresses are determined, the section forces are

$$\widetilde{\boldsymbol{F}}_s = \int_A \mathbf{T}_{ms}^{\mathrm{T}} \cdot \sigma \, dA = \sum_{i=1}^{numFibres} \begin{Bmatrix} 1 \\ -z \end{Bmatrix} \cdot \sigma_i \tag{9}$$

When numerical integration is employed to integrate the section forces to get the basic forces, i.e., to evaluate $\widetilde{\boldsymbol{F}}_b = \int_0^L \mathbf{T}_{sb}^{\mathrm{T}} \, \widetilde{\boldsymbol{F}}_s \, dx + \overline{\boldsymbol{F}}_b$, then integration points are usually given along an ξ-axis from –1 to 1. The relationship between x and ξ is $x=(\xi+1)L/2$,

$$x = \frac{(\xi+1)\cdot L}{2} \tag{10}$$

The displacement-based element is straightforward to implement. Its weakness is that the curvature varies linearly along the element. That is because curvature equals $M(x)/EI$. In light of Eq. (2) and the relationship $M=EIw''(x)$, that curvature is linear. In reality, yielding is often localized near the member ends, causing nonlinearly increasing curvature towards the member ends. For this reason, a member must usually be discretized into several displacement-based elements to get accurate results.

Force Interpolation

The force-based element remedies the problem with linearly varying curvature in the displacement-based element. Instead of saying something about the displacements, and satisfying equilibrium only in an average sense, equilibrium is now enforced. Instead of relying on the displacement relationship

$$\mathbf{u}_s = \mathbf{T}_{sb}\mathbf{u}_b \tag{11}$$

in the right-hand side of Figure 1, attention is given to the force relationship

$$\tilde{\mathbf{F}}_s = \tilde{\mathbf{T}}_{sb}\tilde{\mathbf{F}}_b \tag{12}$$

which does not appear in Figure 1. That poses a problem in the state determination. A detour in Figure 1 is required to circumvent Eq. (11), i.e., to obtain section forces, $\mathbf{F}_s$, for a given section deformations, $\mathbf{u}_s$, which is the aim of the state determination. Figure 3 shows that detour, which entails compatibility iterations in the element in order to determine the basic forces that correspond to the input trial displacements. For every such iteration, the element sends trial element forces to the cross-section, which performs equilibrium iterations in order to determine the section deformations that correspond to the input trial forces. For every such iteration, the section sends trial strain to the uniaxial material at each fibre, i.e., at each integration point of the cross-section.

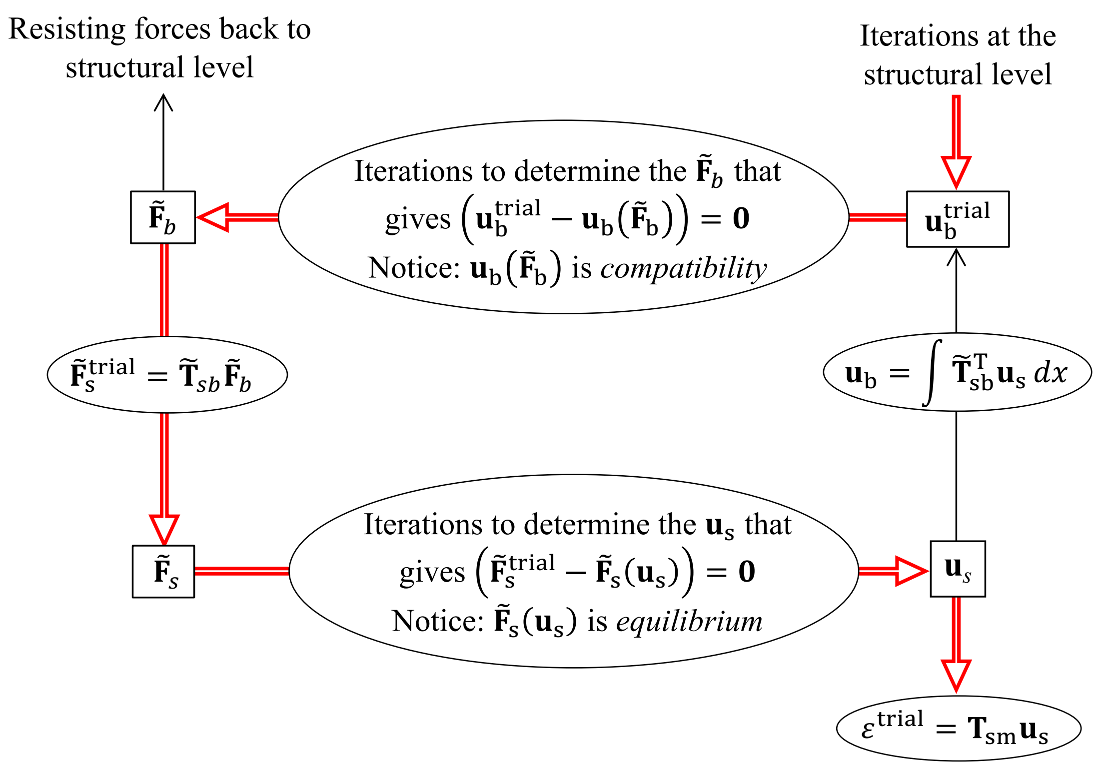

Figure 3: Iterations in a force-based element.

The matrix $\tilde{\mathbf{T}}_{sb}$ in Eq. (12) is obtained by equilibrium:

$$\tilde{\boldsymbol{F}}_{\boldsymbol{s}} = \tilde{\boldsymbol{T}}_{\boldsymbol{sb}} \tilde{\boldsymbol{F}}_{\boldsymbol{b}} \leftrightarrow \left\{ \begin{array}{c} N(x) \\ M(x) \end{array} \right\} = \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & -\left(1-\frac{x}{L}\right) & \frac{x}{L} \end{array} \right] \left\{ \begin{array}{c} N \\ M_1 \\ M_2 \end{array} \right\} \qquad (13)$$

where N=axial force and M=bending moment along the element. Common to the displacement-based and forced-based elements is the transformation from the material-level to the section-level, addressed earlier in this document.

Berkeley Notation

The finite element method has a long history at the University of California at Berkeley. Those pioneering efforts were continued in the development of nonlinear analysis. For that reason, a number of important publications employ the Berkeley-specific notation. That notation for nonlinear frame elements is summarized in Figure 4. Notice that the Local element configuration is omitted; the transformation matrix **a** links the Basic configuration with the Global element configuration. Also, notice that the symbol **b** is employed for the transformation matrix $\tilde{\mathbf{T}}_{sb}$ utilized earlier in this document for force-based elements.

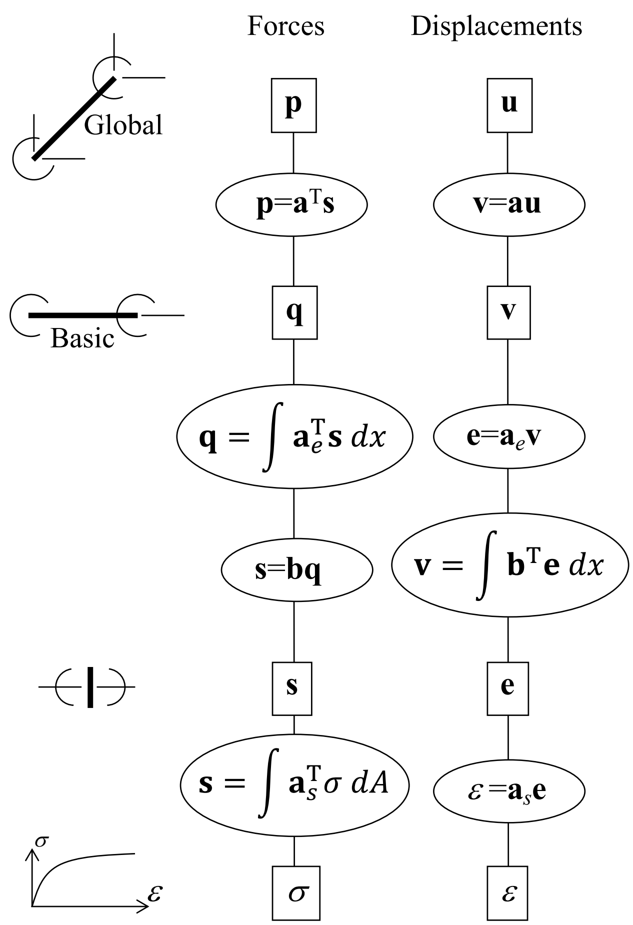

Figure 4: Berkeley notation for nonlinear frame elements.