Continuum Mechanics with Tensors

In other documents posted near this one, equations are provided, often in Voight notation, for the three ingredients of boundary value problems in solid mechanics: kinematic compatibility, material law, and equilibrium. So, why the need for the present document? Because there exists an alternative language, of unparalleled beauty and generality, making extensive use of tensors and index notation, often labelled continuum mechanics. If you had trouble seeing applications when you learned about divergence, flux, gradient, etc. in vector calculus, then your need is here met. While the phrase Theory of Elasticity can be applied to the equations in the aforementioned documents, the label Continuum Mechanics is here used as code word for the use of tensor products, divergence theorems, and all that good stuff. There is substantial overlap in substance, but some of the language differs. Among the books of greatest value and pedagogical rigour in this field is the one by Keith Hjelmstad entitled Fundamentals of Structural Mechanics; two editions are now available. In line with the jargon of continuum mechanics, you will see that he employs the phrase Geometry of Deformation for kinematic compatibility, Constitutive Equations for material law, etc. Let's dive into that world.

Index Notation

First, we refresh our knowledge of index notation, covered in the Math section of the Basics page. There are free and dummy indices. Consider the system of equations $\mathbf{Ax}=\mathbf{b}$, where $\mathbf{A}$ is a matrix, while $\mathbf{x}$ and $\mathbf{b}$ are vectors. In index notation, it is written $A_{ij}x_j=b_i$. There are two ways to tell that i is a free index and that j is a dummy index, in this case. One way is to look at the right-side. It contains solely the vector $\mathbf{b}$. The index i counts over the size of the vector; hence, i is a free index. However, the other way is general and much better: The index i appears only once in both terms of the equation; hence, it is a free index. This rule stems from the fact that summation is implied over repeated indices: $A_{ij}x_j$ should be read $\sum_{j=1}^{N} A_{ij}x_j$, making j a dummy index.

Tensors

A vector is a one-dimensional tensor; a matrix is a two-dimensional tensor. This means that the concept of a tensor is most general. In the same way that we recognize bi as a vector and A_{ij} as a matrix, we recognize C_{ijk} as a third-order tensor, C_{ijkl} as a fourth-order tensor, etc., here arbitrarily named $\mathbf{C}$. Often, the indices of a tensor count over the number of geometry axes of the problem. In basic mechanics, the axes of a 3D problem are labelled x, y, and z. In general, we label them 1, 2, and 3, implying that each index can take the value 1, 2, and 3. Many tensors "live" in such coordinate systems, which is highlighted by introducing the "base vectors" of the coordinate system: $\mathbf{e}_1=\{1, 0, 0\}$, $\mathbf{e}_2=\{0, 1, 0\}$, $\mathbf{e}_3=\{0, 0, 1\}$. This means the base vectors are parallel to their respective axis direction. The "tensor product" $\mathbf{e}_i\otimes\mathbf{e}_j$ is the dyadic product of the two vectors, meaning that it creates a matrix containing all possible products between the components of $\mathbf{e}_i$ and

$\mathbf{e}_j$. Using that definition, the components of a tensor can be given meaning by relating them to the base vectors:

$$\mathbf{A} = \sum_{i=1}^{3}\sum_{j=1}^{3} A_{ij}[\mathbf{e}_i \otimes \mathbf{e}_i] \tag{1}$$

As Hjelmstad says in his book: A tensor does not depend upon the coordinate system, but its components do. Another way to express the same thing is $A_{ij}=\mathbf{e}_i(\mathbf{A}\mathbf{e}_j)$, with inner products implied.

Kinematic Compatibility

The objective of kinematic compatibility equations is to express strains in terms of displacements. Surprisingly, there exists no unique way of expressing strain. It is instructive to start with a uniaxial member of original length L_o, with new length L after deformation. The concept of "stretch" is defined by

$$\lambda = \frac{L}{L_o} \tag{2}$$

Conversely, the engineering strain is "change in length" over "original length:"

$$\varepsilon = \frac{L - L_o}{L_o} \tag{3}$$

which means that $\varepsilon=\lambda-1$. Yet another strain measure for this basic case is the Lagrangian strain, which is similar to the engineering strain, but using squared lengths:

$$E = \frac{1}{2}\left(\frac{L^2 - L_o^2}{L_o^2}\right) \tag{4}$$

The reason for the factor ½ is that it makes E approach ε for small strains, because $E=\frac{1}{2}(\lambda^2-1)=\frac{1}{2}(\lambda-1)(\lambda+1)$, and that last parenthesis is close to 2 for small strains, i.e., when λ is close to unity. In contrast to the Lagrange strain, the Eulerian strain, not spelled out here, employs L instead of L_o in the denominator. The Lagrange strain is central in the solid mechanics of continuum mechanics, in which the displacements of any point within the structural member are described by a "deformation map." That map, denoted by the symbol ϕ, tells where a point located at $\mathbf{X}$ in the undeformed body is found at location $\mathbf{x}$ in the deformed body:

$$\boldsymbol{x} = \phi(\mathbf{X}) \tag{5}$$

In the second edition of his book, Hjelmstad switched to $\mathbf{z}$ and $\mathbf{x}$ for the point in the undeformed and deformed body, respectively, instead of $\mathbf{X}$ and $\mathbf{x}$, which is maintained here. That book contains a pedagogically appealing explanation of how the unit-length tangent vector $d\mathbf{X}/ds$ to a line denoted by s becomes the tangent vector $d\mathbf{x}/ds$ to the deformed line. The chain rule of differentiation yields

$$\frac{d\mathbf{x}}{ds} = \frac{d\mathbf{x}}{d\mathbf{X}} \cdot \frac{d\mathbf{X}}{ds} \equiv \mathbf{F} \cdot \frac{d\mathbf{X}}{ds} \tag{6}$$

where the "deformation gradient tensor" $\mathbf{F}=d\mathbf{x}/d\mathbf{X}$ is defined. In the literature, Eq. (6) is often written simply $d\mathbf{x}=\mathbf{F}d\mathbf{X}$. The tangent vector $d\mathbf{x}/ds$ no longer has unit length; by calculating its deformed length we have the stretch:

$$\lambda = \left\| \frac{d\mathbf{x}}{ds} \right\| = \sqrt{\frac{d\mathbf{x}}{ds} \cdot \frac{d\mathbf{x}}{ds}} \tag{7}$$

In structural engineering it is common to work with the displacements $\mathbf{u}(\mathbf{X})=\{u, v, w\}= \phi(\mathbf{X}) - \mathbf{X}$ instead of $\mathbf{x}$, where $\mathbf{u}$ must not be confused with the vector of degrees of freedom appearing elsewhere on this website. The deformation map is then $\phi(\mathbf{X})=\mathbf{X}+\mathbf{u}(\mathbf{X})$ and, as a result, the deformation gradient tensor reads

$$\mathbf{F} = \mathbf{I} + \nabla \mathbf{u} = \delta_{ij} + u_{i,j} \tag{8}$$

where δ is Dirac's delta and $u_{i,j}$ means du_i/dX_j. The deformation gradient tensor, $\mathbf{F}$, contains information about axial and shear strain at point $\mathbf{X}$. In fact, the stretch is obtained from it by substituting Eq. (6) into Eq. (7), now denoting $d\mathbf{X}/ds$ by $\mathbf{n}$ in order to highlight that it can be any unit vector in the undeformed configuration:

$$\lambda = \sqrt{(\mathbf{F}\cdot\mathbf{n})\cdot(\mathbf{F}\cdot\mathbf{n})} = \sqrt{\mathbf{n}\cdot\mathbf{F}^{\mathrm{T}}\mathbf{F}\,\mathbf{n}} \tag{9}$$

That is a justification for why the "Green deformation tensor" is defined as

$$\mathbf{C} = \mathbf{F}^{\mathrm{T}}\mathbf{F} \tag{10}$$

In order to produce a strain tensor, Eq. (9) rewritten in terms of the definition in Eq. (10) is substituted into the previously stated expression for Lagrange strain, i.e., $E=\frac{1}{2}(\lambda^2-1)$:

$$E = \frac{1}{2}(\mathbf{n}\cdot\mathbf{C}\mathbf{n} - 1) \equiv \mathbf{n}\cdot\mathbf{E}\mathbf{n} \tag{11}$$

where the Lagrangian strain tensor is defined:

$$\mathbf{E} = \frac{1}{2}[\mathbf{C} - \mathbf{I}] \tag{12}$$

Substitution of Eqs. (10) and (8) into Eq. (12) and switching to index notation yields

$$E_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i} + u_{k,i}u_{k,j}\right) \tag{13}$$

That expression is employed in the document entitled Green-Lagrange Strain, posted near this one.

References

Hjelmstad, K. D. (1997). "Fundamentals of Structural Mechanics." Prentice Hall.

De Borst, **Crisfield**, Remmers, Verhoosel (2012). "Nonlinear Finite Element Analysis of Solids and Structures." Second Edition. Wiley.