Beams on Elastic Foundation

Beams on elastic foundation, such as that in Figure 1, appear in building foundations, floating structures, beams that rest on a grid of perpendicular beams, and elsewhere. It also turns out that the governing differential equation is similar to that of cylindrical shells and tapered beams with curved webs. As a result, this material has quite broad applicability. However, it is important to exercise caution when applying this theory to problems where the foundation does not behave in a linear fashion, such as soils that respond nonlinearly to pressure or ships whose waterline area changes with sinkage.

In this document the beam is assumed to be an ordinary Euler-Bernoulli beam, thus the theory from that document carries over to this one. Accordingly, downwards distributed load on the beam is referred to as q, while it is denoted q_z when it acts in the opposite direction, i.e., in the z-direction.
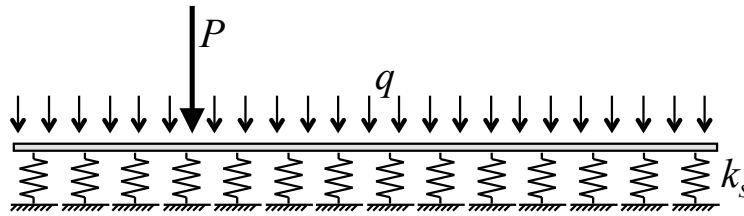

Figure 1: Beam on elastic foundation.

Foundation Stiffness

The foundation stiffness, k_s, which is illustrated in Figure 1, is conceptually straightforward. When the beam displaces downwards, the foundation exerts an upwards force, illustrated by the springs in Figure 1. k_s has units of force per unit length along the beam. Another stiffness, denoted k_ϕ, is also present, relating to rotation ϕ of the beam cross-section around the longitudinal axis of the beam. In other words, k_ϕ provides resistance against torsion of the beam. Techniques to determine k_s and k_ϕ are described in the following subsections for different situations.

Stiffness from Buoyancy

An important application of this theory is floating structures. To determine k_s for a beam that floats in water, suppose the beam cross-section has width equal to b at the location where the cross-section penetrates the water surface. Furthermore, let the mass density of water be denoted ρ_w, which typically equals 1,025kg/m³ for seawater, and let g denoted the acceleration of gravity. For an infinitesimal beam length, dx, the change in buoyancy force due to a vertical displacement Δ equals the weight of the displaced water, namely weight density multiplied by volume:

$$B = \left(\rho_w \cdot g \right) \cdot \left(b \cdot dx \cdot \Delta \right) \tag{1}$$

As a result, the stiffness that resists the displacement Δ, per unit length of the beam, is obtained by dividing B by dx and Δ:

$$k_s = \rho_w \cdot g \cdot b \left[\frac{kN}{m^2} \right] \tag{2}$$

which has unit force per unit length squared. The evaluation of k_s according to Eq. (2) is correct as long as the beam width at the waterline does not change. Unfortunately, this condition is not always satisfied, which is understood from Figure 2. While the rectangular beam-cross-section maintains width b regardless of the vertical displacement, the value of b for the circular cross-section varies with the heave motion. As a result, k_s is a function of Δ for the circular cross-section, which introduces nonlinearity that is neglected in the subsequent equations.
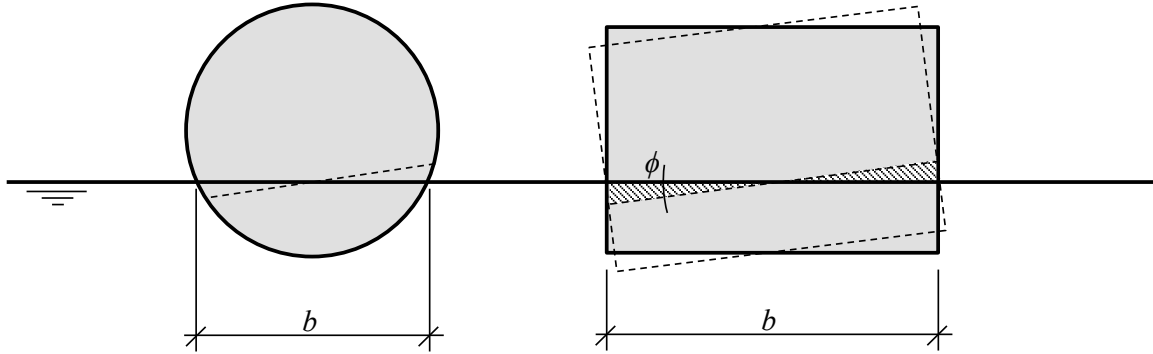
Figure 2: Heave and roll motion of floating beams.

The fact that the waterline area is only a proxy for displaced water volume becomes even more apparent when determining k_ϕ. To understand this, first consider the classical approach for determining k_ϕ. When the rectangular beam-cross-section in Figure 2 rotates counterclockwise as indicated by dashed lines, more water is displaced on the left side and less water is displaced on the right side. As a result, buoyancy forces act against the rotation, giving rise to k_ϕ. By computing the volumes of the shaded triangles in Figure 2 the resultant buoyancy force, on either side, is

$$R = \left(\rho_w \cdot g \right) \cdot \frac{1}{2} \cdot \frac{b}{2} \cdot \frac{b}{2} \cdot \phi \cdot dx \tag{3}$$

The distance between the force pair, i.e., the buoyancy forces from each of the shaded triangles in Figure 2 is 2/3 of b. Thus, the moment from buoyancy that resists the rotation is

$$T = R \cdot \frac{2}{3} \cdot b = \rho_w \cdot g \cdot \frac{b^3}{12} \cdot \phi \cdot dx \tag{4}$$

As a result, the stiffness that resists the rotation ϕ, per unit length of the beam, is obtained by dividing T by dx and ϕ:

$$k_\phi = \rho_w \cdot g \cdot \frac{b^3}{12} \tag{5}$$

where it is observed that the moment of inertia of the waterline area. Again it is observed that the derivations do not hold for the circular cross-section in Figure 2. In fact, it is

easily seen that cross-section has zero resistance against rotation. If not easily seen, it is easily felt when trying to balance on a floating timber log. The problem with applying Eq. (5) to the circular cross-section is the use of movement of the beam at the waterline as a proxy for displaced water volume. Thus, while Eq. (5) is commonly used, it is important to be aware of its limitations.

Stiffness from Soil

Suppose the stiffness, k_s, is determined from soil testing. In particular, suppose a vertical load, P, is placed on an area with dimensions x and y, and that the vertical displacement, Δ, is measured. The relationship between the distributed load and the displacement is written in terms of a distributed stiffness, k_d:

$$\frac{P}{x \cdot y} = k_d \cdot \Delta \quad \Rightarrow \quad k_d = \frac{P}{\Delta \cdot x \cdot y}\left[\frac{N}{m^3}\right] \tag{6}$$

While k_d is stiffness per unit area, k_s is stiffness per unit length. The sought value is obtained by multiplying k_d by the beam width, b:

$$k_s = b \cdot k_d \left[\frac{N}{m^2}\right] \tag{7}$$

If one assumes that the foundation material is linear elastic, there is no unique relationship between the Young's modulus, E, of the foundation material and the stiffness k_s. However, if one imagines that the soil underneath the beam is linear elastic with depth L to bedrock then the force-deformation relationship of the soil is

$$P = \frac{EA}{L} \cdot \Delta \tag{8}$$

where $A=xy$ is the area loaded by P. Writing Eq. (8) in the form of Eq. (6) yields

$$\frac{P}{x \cdot y} = k_d \cdot \Delta \quad \Rightarrow \quad k_d = \frac{E}{L}\left[\frac{N}{m^3}\right] \tag{9}$$

and the sought stiffness is, in accordance with Eq. (7):

$$k_s = b \cdot k_d = \frac{b \cdot E}{L}\left[\frac{N}{m^2}\right] \tag{10}$$

Stiffness from Girder Grid

Another situation appears when the beam is resting on a grid of closely spaced perpendicular beams, e.g. joists. Suppose the joists are spaced at x on centre and that their stiffness against vertical deflection at the point of intersection with the beam is k_b[N/m]. Then the sought stiffness is:

$$k_s = \frac{k_b}{x}\left[\frac{N}{m^2}\right] \tag{11}$$

Stiffness from Cylindrical Shell Wall

The document on cylindrical shells, posted on this website ends up with a differential equation of the form

$$\frac{EI}{1-v^2} \cdot \frac{d^4w}{dx^4} + \frac{E \cdot t}{r^2} \cdot w = p \tag{12}$$

where t=wall thickness, r=cylinder radius, and p= pressure on the cylinder wall. That pressure is essentially resisted by bending stiffness, $EI/(1-v^2)$, of a beam that forms a strip along the cylinder wall, which is the first term in Eq. (12), as well as the hoop stress effect, represented by the stiffness Et/r^2 in the second term in Eq. (12). In other words, the beam that forms a strip along the cylinder wall is a beam on elastic foundation, with stiffness

$$k_s = \frac{E \cdot t}{r^2} \tag{13}$$

"Stiffness" from Ponding

Ponding is the phenomenon of progressive load concentration as the weight of water bending a beam, leading to concentration of water where there is most deflection. In this case, the differential equation is the same as for a beam on elastic foundation, except the stiffness of that foundation is now negative:

$$k_s = -\rho_w \cdot g \cdot b \tag{14}$$

where ρ_w=mass density of water, g=9.81m/s^2, and b=width of the beam considered, e.g., unit width. The negative stiffness is a unique situation in this document, not addressed by the general solution to the differential equation for a beam on elastic foundation established later in this document. However, a separate example on ponding posted near this document addresses that unique case.

Bending

Differential Equation

Compared with the basic Euler-Bernoulli beam theory, it is sufficient to modify the equation for vertical equilibrium to obtain the differential equation for a beam on elastic foundation. As a result, the following conventions from basic beam bending hold valid: 1) Clockwise shear force is positive; 2) Bending moment with tension at the bottom is positive; 3) Tension stress is positive; 4) The z-axis points upwards, so that upwards displacement, w, is positive; 5) The distributed load, q_z, is positive upwards. Figure 3 shows the forces acting on an infinitesimal beam element. The springs that illustrate the elastic foundation exert a downward force when the beam is subjected to an upward displacement.
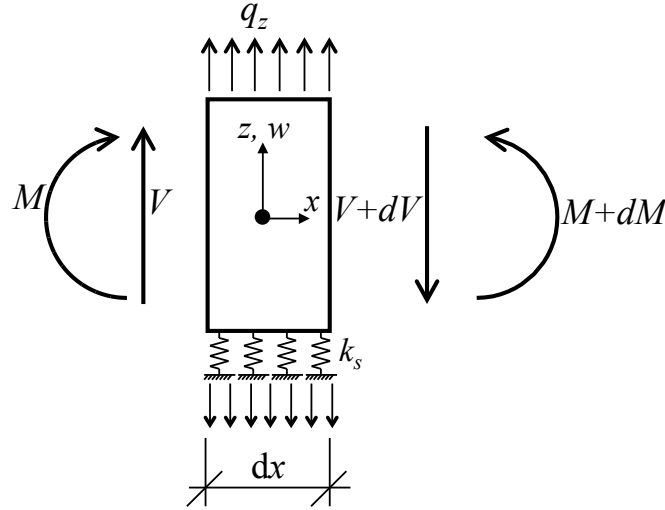
Figure 3: Infinitesimal beam element.

Vertical equilibrium yields:

$$q_z \cdot dx - k_s \cdot w \cdot dx - dV = 0 \tag{15}$$

Dividing by *dx* and re-arranging yields

$$q_z = \frac{dV}{dx} + k_s \cdot w \tag{16}$$

Substitution of this equation into the Euler-Bernoulli beam theory yields the following revised differential equation

$$EI \cdot \frac{d^4 w}{dx^4} + k_s \cdot w = q_z \tag{17}$$

Another way of deriving the differential equation is to start with the following basic differential equation for beam bending:

$$EI \cdot \frac{d^4 w}{dx^4} = q_z \tag{18}$$

From earlier it is understood that the applied load, q_z, plus the elastic foundation yields a total force on the beam element equal to q_z–$k_s w$. By substituting this total load in the right-hand side of Eq. (18), Eq. (17) is obtained. For convenience, it is rewritten on the form

$$\frac{d^4 w}{dx^4} + \frac{k_s}{EI} \cdot w = \frac{q_z}{EI} \tag{19}$$

In solving this differential equation, it is useful to define a "characteristic length," sometimes referred to as the elastic length. To approach the definition, it is first noted that EI/k_s has dimension [m^4]. As a result, the following definition of the characteristic length has the dimension of length:

$$l_c \equiv \sqrt[4]{\frac{4 \cdot EI}{k_s}} \quad \Rightarrow \quad \frac{EI}{k_s} = \frac{l_c^4}{4} \tag{20}$$

The convenience of the factor 4 will become apparent later. It is also convenient to work with the normalized coordinate ξ instead of the original coordinate, x, along the beam:

$$\xi = \frac{x}{l_c} \tag{21}$$

To transform the differential equation, differentiation with respect to x is related to differentiation with respect to ξ by the chain rule of differentiation:

$$\frac{d}{dx} = \frac{d}{d\xi} \cdot \frac{d\xi}{dx} = \frac{1}{l_c} \cdot \frac{d}{d\xi} \tag{22}$$

Hence, the fourth-order derivative with respect to x equals the fourth-order derivative with respect to ξ divided by l_c^4, which yields the following homogeneous version of the transformed differential equation, now formulated in the coordinate ξ:

$$\frac{d^4 w}{d\xi^4} + 4 \cdot w = 0 \tag{23}$$

General Solution

The characteristic equation is $\gamma^4 + 4 = 0$ has the four different complex roots $(1+i)$, $(1-i)$, $(-1+i)$, and $(-1-i)$. Consequently, the general solution is:

$$w(\xi) = \underbrace{C_1 \cdot e^{-\xi} \cos(\xi) + C_2 \cdot e^{-\xi} \sin(\xi)}_{\text{"Damped terms"}} + \underbrace{C_3 \cdot e^{\xi} \cos(\xi) + C_4 \cdot e^{\xi} \sin(\xi)}_{\text{"Undamped terms"}} \tag{24}$$

where the phrase "damped terms" is employed to identify terms that vanish as ξ increases to infinity. This labeling is useful because the solution for a point load must vanish far away from the point of load application. In fact, only the damped terms appear in many practical situations. To shorten the notation under such circumstances, the following auxiliary functions are defined:

$$g_1 = e^{-\xi} \cos(\xi) \tag{25}$$

$$g_2 = e^{-\xi} \sin(\xi) \tag{26}$$

$$g_3 = g_1 + g_2 \tag{27}$$

$$g_4 = g_1 - g_2 \tag{28}$$

These functions are plotted in Figure 4, where it is observed that they decay rapidly with ξ. In fact, all functions approach zero once ξ increases beyond 4.
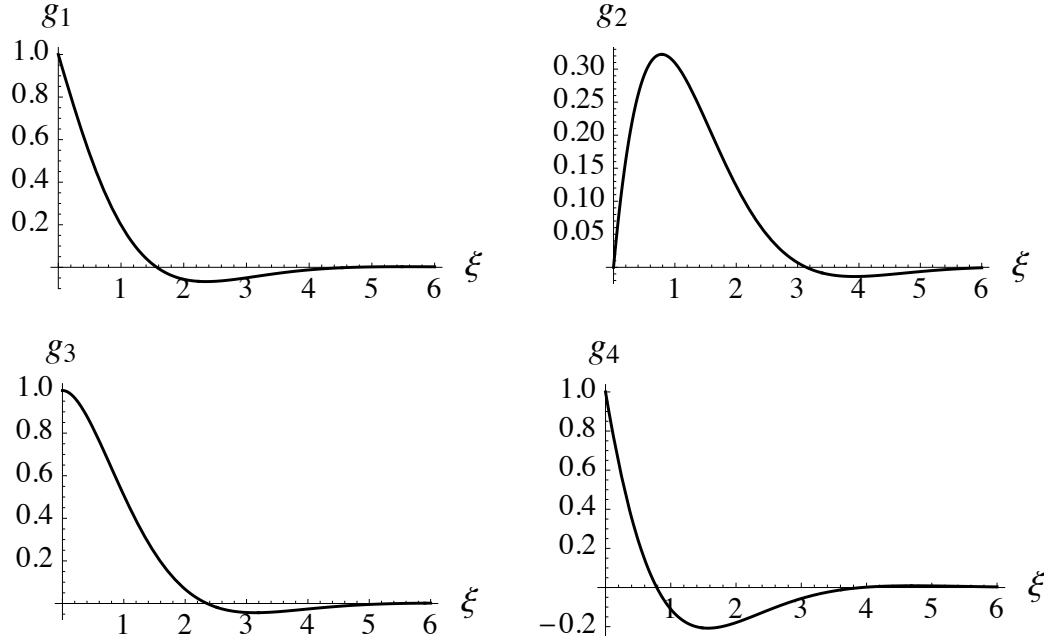
Figure 4: g-functions.

The auxiliary functions also have the following properties:

$$\frac{\mathrm{d}g_1}{\mathrm{d}\xi} = -g_3 \qquad \frac{\mathrm{d}g_2}{\mathrm{d}\xi} = g_4 \qquad \frac{\mathrm{d}g_3}{\mathrm{d}\xi} = -2g_2 \qquad \frac{\mathrm{d}g_4}{\mathrm{d}\xi} = -2g_1 \tag{29}$$

These relationships are helpful for deriving the bending moment, etc. from the general solution. The starting point is the general solution with only the damped terms, which reads:

$$w(\xi) = C_1 \cdot g_1 + C_2 \cdot g_2 \tag{30}$$

Differentiation yields the rotation:

$$\theta(\xi) = \frac{dw}{dx} = \frac{1}{l_c} \cdot \frac{dw}{d\xi} = -\frac{1}{l_c} \cdot \left(C_1 \cdot g_3 - C_2 \cdot g_4 \right) \tag{31}$$

Another differentiation yields the bending moment:

$$M = EI \cdot \frac{d^2 w}{dx^2} = EI \cdot \frac{d\theta}{dx} = EI \cdot \frac{1}{l_c} \cdot \frac{d\theta}{d\xi} = \frac{2EI}{l_c^2} \cdot \left(C_1 \cdot g_2 - C_2 \cdot g_1 \right) \tag{32}$$

Yet another differentiation yields the shear force:

$$V = \frac{\mathrm{d}M}{\mathrm{d}x} = \frac{1}{l_c} \cdot \frac{\mathrm{d}M}{\mathrm{d}\xi} = \frac{2EI}{l_c^3} \cdot \left(C_1 \cdot g_4 + C_2 \cdot g_3 \right) \tag{33}$$

Beam with End Forces

As a "reference case" for the analysis of several other problems it is useful to consider the beam shown in Figure 5, where one end is subjected to the forces V_o and M_o, while the other end is "infinitely" far away.
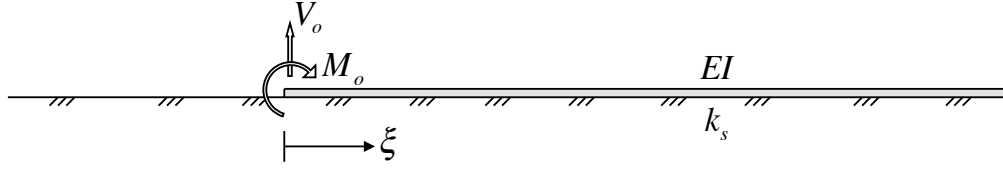

Figure 5: Beam with end-forces.

Notice that both V_o and M_o are positive, i.e., the shear force is clockwise and the bending moment gives tension at the bottom. Because the beam is infinitely long in one direction, the solution cannot have contributions from the un-damped terms. As a result, the solution is given by Eq. (30). Because $g_1=g_3=g_4=1$ and $g_2=0$ at $\xi=0$, Eqs. (32) and (33) yield the follow expression for the bending moment and shear force at $\xi=0$:

$$M(\xi = 0) = -\frac{2EI}{l_c^2} C_2 \tag{34}$$

$$V(x = 0) = \frac{2EI}{l_c^3}\left(C_1 + C_2\right) \tag{35}$$

Setting those expressions equal to M_o and V_o, respectively, yields:

$$C_1 = \frac{l_c^2}{2EI}\left(l_c V_o + M_o\right) \tag{36}$$

$$C_2 = -\frac{l_c^2}{2EI} M_o \tag{37}$$

Substitution of Eqs. (36) and (37) into Eq. (30) to (33) yields the following solution for this loading case:

$$w = \frac{l_c^2}{2EI}\left(M_o g_4 + l_c V_o g_1\right) \tag{38}$$

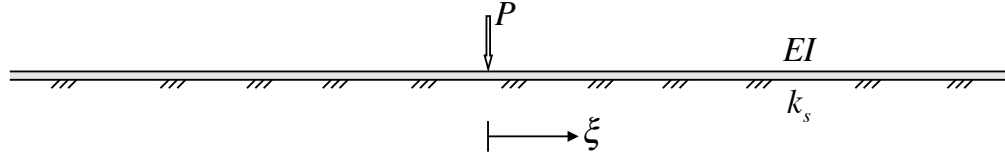$$\theta = -\frac{l_c}{2EI}\left(2M_o g_1 + l_c V_o g_3\right) \tag{39}$$

$$M = M_o g_3 + l_c V_o g_2 \tag{40}$$

$$V = -\frac{2}{l_c} M_o g_2 + V_o g_4 \tag{41}$$

Beam with Point Load

The previous solution is now utilized to analyze the infinitely long beam in Figure 6, which has a point load applied at $\xi=0$.

Figure 6: Beam with point load.

Immediately to the left of the point load the shear force is P/2, while immediately to the right it is –P/2. That is, with reference to the previous case, $V_o = -P/2$. Furthermore, the rotation at the point load is zero:

$$\theta(0) = \frac{l_c}{2EI}\left(2M_o + l_c V_o\right) = 0 \tag{42}$$

Substituting $V_o = -P/2$ and solving for M_o yields

$$M_o = \frac{P \cdot l_c}{4} \tag{43}$$

where it is noted that the bending moment at the point load is the same as that of a simply supported beam with length l_c loaded at midspan. In fact, that is the reason that the integer 4 was introduced in the definition of l_c in Eq. (20). In short, the solution for the beam in Figure 6 is given by Eqs. (38) to (41) with

$$V_o = -\frac{P}{2} \tag{44}$$

and

$$M_o = \frac{l_c P}{4} \tag{45}$$

That yields the following solution for a beam on an elastic foundation with a point load:

$$w = \frac{Pl_c^3}{8EI}\left(g_4 - 2g_1\right) \tag{46}$$

$$\theta = -\frac{Pl_c^2}{4EI}\left(g_1 - g_3\right) \tag{47}$$

$$M = \frac{Pl_c}{4}\left(g_3 - 2g_2\right) \tag{48}$$

$$V = -\frac{P}{2} \cdot \left(g_2 + g_4\right) \tag{49}$$

This solution, as the previous ones, is defined only for positive ξ-values. However, it is readily plotted on both sides of the origin by multiplying Eqs. (47) and (49) by the sign of ξ, namely ξ/|ξ|. This is done because the rotation and the shear force are asymmetric functions. For example, from the discussion before Eq. (42) it is clear that the shear force

is positive where ξ is negative, and vice versa. Another common plotting convention is to multiply Eq. (48) by (–1) to get the moment diagram on the tension side of the beam.

Cluster of Point Loads

If more than one point load is present, such as the situation in Figure 7, the solution is obtained by superposition. Assuming that the beam is sufficiently long for the solution to "dampen out" before the beam end is reached, the solution in Eqs. (46) to (49) is employed. For example, for the situation in Figure 7 the bending moment is:

$$M = \frac{P_0 \cdot l_c}{4}\left(g_3(\xi) - 2g_2(\xi)\right) + \sum_{i=1}^{2} \frac{P_i \cdot l_c}{4}\left(g_3(\xi - \xi_i) - 2g_2(\xi - \xi_i)\right) \tag{50}$$
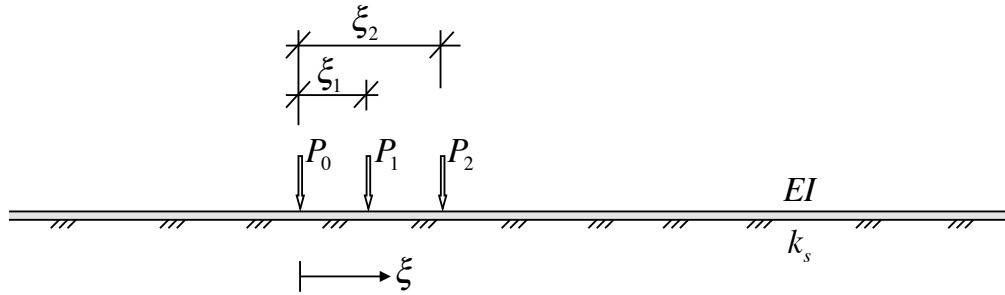

Figure 7: Beam with point load cluster.

Distributed Load

Above, the solutions w, θ, M, and V were established for a beam with a point load. That solution can also be employed in the analysis of beams with distributed load. The point load that was previously applied at $\xi=0$ is now applied at the generic location $\xi=c$. Moreover, the point load, P, is considered to be one infinitesimal part of a distributed load, i.e., $P=q \cdot dx$. Because $dx=l_c \cdot d\xi$ the load to be inserted into the previous solutions $w(\xi)$, $\theta(\xi)$, $M(\xi)$, and $V(\xi)$ is $P=q \cdot l_c \cdot d\xi$. On that basis, the solution for a distributed load between $\xi=a$ and $\xi=b$ is

$$w(\xi) = \int_a^b w(\xi - c)\, dc = -\frac{q l_c^3}{8EI} \cdot l_c \cdot \int_a^b \left(g_4(\xi - c) - 2g_1(\xi - c)\right) dc \tag{51}$$

etc. for the $\theta(\xi)$, $M(\xi)$, and $V(\xi)$, assuming constant load intensity, q.

Torsion

Differential Equation

The differential equation for St. Venant torsion, without resistance from an elastic foundation, is

$$GJ \cdot \phi'' + m_x = 0 \tag{52}$$

The torsional resistance from an elastic foundation is a distributed torque equal to $k_\phi \cdot \phi$, which implies that the effective distributed torque is equal to $m_x - k_\phi \cdot \phi$. That leads to the following modified differential equation for St. Venant torsion:

$$\phi '' - \frac{k_\phi}{GJ} \cdot \phi + \frac{m_x}{GJ} = 0 \tag{53}$$

Now consider the homogeneous version of this equation, i.e., let $m_x=0$, and define the auxiliary quantity

$$l_c = \sqrt{\frac{GJ}{k_\phi}} \tag{54}$$

which is the characteristic length in torsion. This definition gives a length-measure because k_ϕ is in units of force. As done previously for bending, the normalized coordinate ξ is defined as in Eq. (21): $\xi = x/l_c$, with resulting differentials given in Eq. (22). This means that the homogeneous version of Eq. (53) is

$$\frac{d^2\phi}{dx^2} - \frac{1}{l_c^2} \cdot \phi = \frac{1}{l_c^2} \cdot \frac{d^2\phi}{d\xi^2} - \frac{1}{l_c^2} \cdot \phi = \frac{d^2\phi}{d\xi^2} - \phi = 0 \tag{55}$$

General Solution

The characteristic equation is $\gamma^2 + \gamma = 0$ has the two roots $\gamma = 0$ and $\gamma = 1$. Consequently, the general solution is:

$$\phi = C_1 \cdot e^\xi + C_2 \cdot e^{-\xi} \tag{56}$$

The torque associated with this solution is

$$T = GJ \cdot \frac{d\phi}{dx} = GJ \cdot \frac{1}{l_c} \cdot \left(C_1 \cdot e^\xi + C_2 \cdot e^{-\xi} \right) \tag{57}$$

Beam with End Torque

As a reference case for the analysis of other problems, consider a beam like the one in Figure 5, with a torque, T_o, applied at the free end. Because the beam is infinitely long in one direction, the solution cannot have contributions from the "un-damped" term. As a result, $C_1 = 0$, and the equation that determines C_2 is:

$$T_o = T(\xi = 0) = GJ \cdot \frac{1}{l_c} \cdot C_2 \tag{58}$$

which yields

$$C_2 = \frac{T_o \cdot l_c}{GJ} \tag{59}$$

which in turn means that the solution is

$$\phi = \frac{T_o \cdot l_c}{GJ} \cdot e^{-\xi} \tag{60}$$

Rigid Beam

For cases with long characteristic length relative to the beam length, L, it may be appropriate to consider the beam as a rigid body as it rotates under applied torque, T_o. In that case the rotation is simply the constant

$$\phi_o = \frac{T_o}{k_\phi \cdot L} \tag{61}$$

and the torque in the beam increases linearly from zero at the free end to

$$T(x) = k_\phi \cdot \phi_o \cdot x = T_o \cdot \frac{x}{L} \tag{62}$$

at a distance x from the free end.