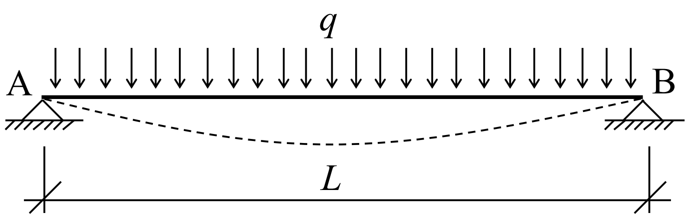
Beam with Membrane Effect

The beam shown below has pin-supports at both ends. That fact, combined with large deformations, implies that the distributed load will be carried by two effects: Beam bending plus the membrane effect, which could also be called the hammock effect. Another name for this phenomenon is geometric nonlinearity.



An energy method is utilized here to solve the problem. That implies using the principle of minimum potential energy (PMPE), which reads

$$\delta\Pi=0$$

where δ means "variation" and Π is the total potential energy, which is

$$\Pi=U+H$$

where U is strain energy and H is the potential energy in the external load. The latter is

$$H = -\int_0^L q\, w\, dx$$

where w is downwards displacement. The strain energy is

$$U = \frac{1}{2} E \int_0^L \int_A \epsilon^2\, dA\, dx$$

where E is Young's modulus. To include the membrane effect it is necessary to use a strain expression that includes geometric nonlinearity. That is included in Green's strain, which is the second term here added to the ordinary strain associated with bending:

$$\epsilon = z \frac{d^2 w}{dx^2} + \frac{1}{2} \left(\frac{dw}{dx}\right)^2$$

That strain is now substituted into the expression for U. Because the area-integral of z is zero, the terms with z will vanish when U is calculated, while those with z^2 give rise to the moment of inertia; hence, the strain energy reads

$$U = \int_0^L \left(\frac{EI}{2} \left(\frac{d^2 w}{dx^2} \right)^2 + \frac{EA}{8} \left(\frac{dw}{dx} \right)^4 \right) dx$$

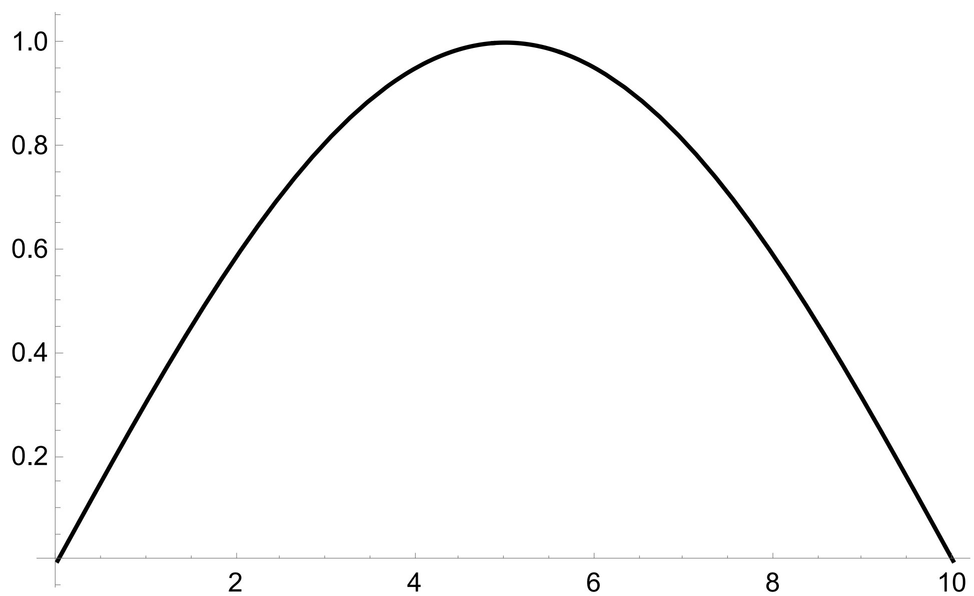
The first term in U is the ordinary contribution from bending; the last term account for the strain from the rotation of the beam segments. Now, consider a trigonometric shape function:

$$w = \triangle \, \mathrm{Sin}\left[\frac{\pi}{L} x \right];$$

Here is a plot of that Ritz function:

```
Plot[w /. {△ -> 1, L -> 10}, {x, 0, 10}, PlotStyle -> Black]
```

which yields:



The shape function is next differentiated and substituted into the strain energy integral:

```
dw = D[w, x];
```

```
ddw = D[dw, x];
```

$$U = \int_0^L \left(\frac{EI}{2} ddw^2 + \frac{EA}{8} dw^4 \right) dx;$$

The potential energy in the load is:

```
H = ∫_0^L -q w dx;
```

The maximum displacement is, from the principle of minimum potential energy:

```
Π = U + H;
equation = D[Π, Δ];
sol = Solve[equation == 0, Δ];
```

That solution looks like this:

```
sol[[1]] // Simplify
```

which yields:

$$\left\{\Delta \to \frac{2}{3}\left(-\frac{2^{2/3}\,\mathrm{EI}\,\pi^{5/3}}{\left(9\,\mathrm{EA}^2\,L^4\,q+\sqrt{\mathrm{EA}^3\left(2\,\mathrm{EI}^3\,\pi^{10}+81\,\mathrm{EA}\,L^8\,q^2\right)}\right)^{1/3}}+\frac{2^{1/3}\left(9\,\mathrm{EA}^2\,L^4\,q+\sqrt{\mathrm{EA}^3\left(2\,\mathrm{EI}^3\,\pi^{10}+81\,\mathrm{EA}\,L^8\,q^2\right)}\right)^{1/3}}{\mathrm{EA}\,\pi^{5/3}}\right)\right\}$$

In order to show numerical results, consider a European wide-flange beam HEB120 spanning 5m and bending about its strong axis:

```
E = 200 000;
values = {EA -> E 3400, EI -> E 8 640 000, L -> 5000};
```

The maximum displacement for a load *q*=3kN/m is:

```
Δ /. sol[[1]] /. values /. q -> 3 // N
```

which yields: 13.7956

Compare that with ordinary beam theory for a simply supported beam:

```
ΔBasic = 5 q L^4 / (384 EI);
ΔBasic /. values /. q -> 3 // N
```

which yields: 14.1285

The ratio of length to displacement is often used as an index to judge if the displacement is acceptable (around 300 to 360 tends to be accepted):

```
(          L
 ─────────────────────── ) /. values
 ΔBasic /. q -> 3 // N
```

which yields: `353.894`

Although it is difficult to imagine how the supports can be designed to have absolutely zero horizontal displacement when the loads come onto the beam, notice how the "hold-back" and the membrane effect increases when the load increases:

```
Plot[{ΔBasic /. values, Δ /. sol[[1]] /. values}, {q, 3, 10},
 AxesLabel -> {"Load", "Displacement"},
 PlotStyle -> {{Black, Dashed}, {Black, Thin}},
 PlotLegends →
  Placed[LineLegend[{"Ordinary beam theory", "With membrane effect"}],
   {Left, Top}]]
```

which yields: