Uniaxial Plasticity Material Model

Plasticity theory addresses materials that yield, and steel is the typical example. Figure 1 shows the idealized stress-strain response of an imagined steel coupon that is first stretched, then unloaded and compressed, then stretched again. The stiffness transitions from the elastic stiffness, E, to the plastic stiffness, αE, whenever the material enter a yielding phase, called "plastic flow." If the material had been perfectly plastic, then the plastic stiffness would have been zero. The plastic flow is the cause of "hardening," which is the first topic addressed in this document.
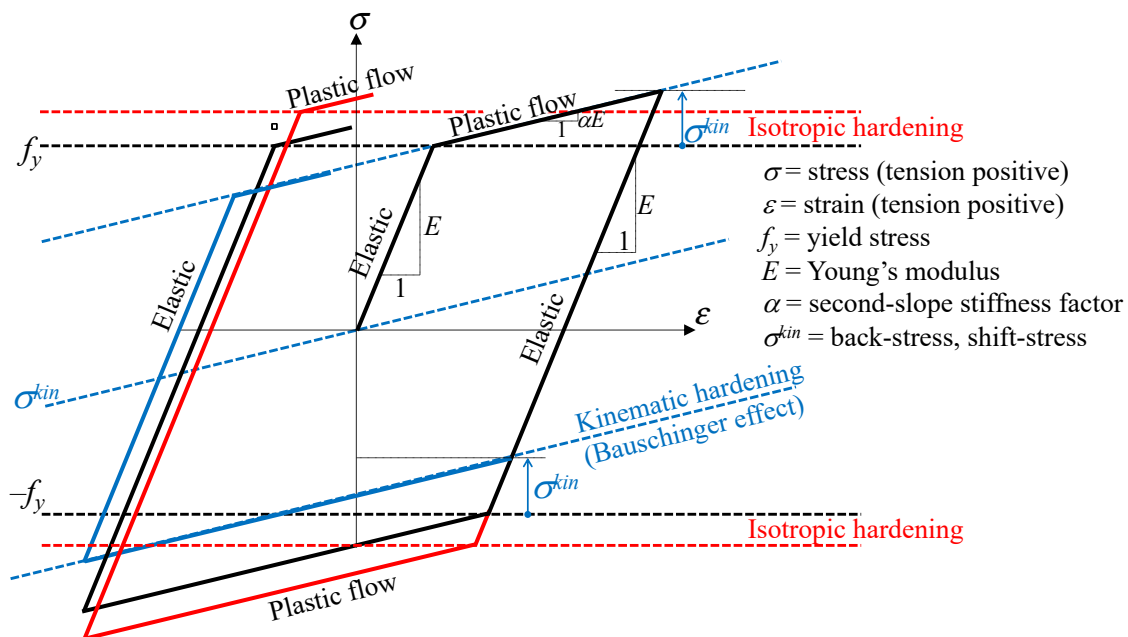

Figure 1: Theoretical uniaxial plastic material behaviour.

Hardening

The black line in Figure 1 describes a situation where the material yields any time the absolute value of the stress, $|\sigma|$, reaches the yield stress, f_y. That is the case of no "hardening." However, plastic flow usually causes some form of hardening. The blue and red lines in Figure 1 exhibits two different forms of hardening:

- **Kinematic hardening** means that the plastic flow follows the two outermost slanted blue dashed lines in Figure 1. Those two lines delineate the stress range within which the response is elastic. The length of that stress range remains $2f_y$, but the elastic stress range shifts by σ^{kin}. In other words, with kinematic hardening for a uniaxial material, the stress range within which the stress remains elastic is $[-f_y+\sigma^{kin}, f_y+\sigma^{kin}]$. Kinematic hardening is a way to model the Bauschinger effect, after the professor of Engineering Mechanics at Munich Polytechnic, Johann Bauschinger (1834–1893).

- **Isotropic hardening** is represented by red lines in Figure 1. Similar to kinematic hardening, the result of isotropic hardening is a change in the stress range within which the response remains elastic. However, in isotropic hardening the stress range does not shift; it changes length.

In the plasticity theory presented next, the region in the stress space within which the response remains elastic is bounded by a "yield surface." Kinematic hardening merely shifts that surface, while isotropic hardening changes the size of the elastic region bounded by that surface.

Second Deviatoric Stress Invariant

The document on stress-based failure criteria introduces the deviatoric stress tensor, **s**, and the associated stress invariants, J_1, J_2, and J_3. It was established that the von Mises failure criterion employs the second deviatoric stress invariant, J_2:

$$\sqrt{3 \cdot J_2} < f_y \tag{1}$$

In general,

$$J_2 = \frac{1}{6} \cdot \left(\sigma_{xx} - \sigma_{yy} \right)^2 + \frac{1}{6} \cdot \left(\sigma_{yy} - \sigma_{zz} \right)^2 + \frac{1}{6} \cdot \left(\sigma_{zz} - \sigma_{xx} \right)^2 + \sigma_{xy}^2 + \sigma_{yz}^2 + \sigma_{zx}^2 \tag{2}$$

For plane stress states,

$$J_2 = \frac{\sigma_{xx}^2}{3} + \frac{\sigma_{yy}^2}{3} - \frac{\sigma_{xx}\sigma_{yy}}{3} + \sigma_{xy}^2 \tag{3}$$

For uniaxial stress,

$$J_2 = \frac{\sigma^2}{3} \tag{4}$$

Because of the formulation in Eq. (1), where the number three appears, Eq. (4) reasonably implies that the von Mises failure criterion for the uniaxial case is $|\sigma| < f_y$. J2 plasticity continues the use of the deviatoric stress. Classical J2 plasticity exhibits a non-smooth transition between the elastic and plastic states, as shown in Figure 1 for the uniaxial case. Generalized J2 plasticity is an extension characterized by a smooth transition between the elastic and plastic response regimes.

Yield Function

The concept of a yield function, or yield condition, whose value is determined by the deviatoric stress, is central in plasticity theory. Denoted by f, the yield function is a function of the deviatoric stress and various material constants. $f < 0$ implies an elastic stress state, while "plastic flow" may occur when the stress state is on the "yield surface" in the stress space, i.e., $f=0$. In generalized plasticity, addressed later, the stress state is allowed to exceed the yield surface. $f > 0$ implies that inelastic effects may be occurring, depending on whether loading or unloading occurs. For the uniaxial stress case illustrated in Figure 1, the **black-line case** of no hardening is represented by the yield function

$$f = |\sigma| - f_y \tag{5}$$

which also represents the von Mises failure criterion in Eq. (1). The **blue-line case** in Figure 1, illustrating kinematic hardening, is represented by the yield function

$$f = |\sigma - \sigma^{kin}| - f_y \tag{6}$$

The **red-line case** in Figure 1, illustrating isotropic hardening, is represented by the yield function

$$f = |\sigma| - \left(f_y + f^{iso}\right) \tag{7}$$

where f^{iso} is the shift of the red line. Mixing kinematic and isotropic hardening leads to the following yield function for uniaxial stress:

$$f = |\sigma - \sigma^{kin}| - \left(f_y + f^{iso}\right) \tag{8}$$

Python

A uniaxial material model with kinematic hardening is implemented in Python and posted on this website under the name G2 Bilinear Material Class.

Formal Model Formulation

The intention of the equations above is to give a sense of what a yield function is, and what kinematic and isotropic hardening represent, especially related to Figure 1. In the following, a more formal plasticity model is presented for the uniaxial case, resting on some of the material that Professor Armero at the University of California at Berkeley taught me at the turn of the century. The following model is "rate independent," meaning that the loading is so slow that the strain rate with respect to actual time is not relevant. The modelling assumptions are:

1. The **yield function** is formulated in terms of an isotropic hardening variable, α, whose value changes during yielding:

$$f = |\sigma - \sigma^{kin}| - \left(f_y + f^{iso}(\alpha)\right) \tag{9}$$

2. The **additive strain decomposition** separates the total strain into an elastic part, ε^e, and a plastic part, ε^p:

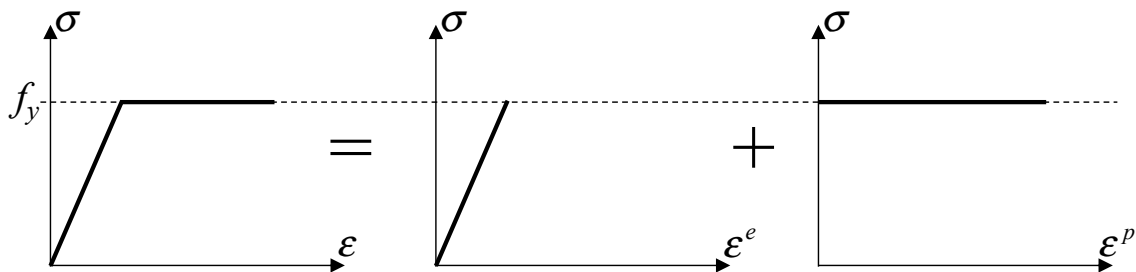$$\varepsilon = \varepsilon^e + \varepsilon^p \tag{10}$$

Figure 2: Relationship between elastic and plastic strain.

3. A "perfectly plastic" **stress-strain relationship** is employed, implying that only the elastic part of the strain causes stress, as illustrated in Figure 2:

$$\sigma = E \cdot \varepsilon^e \tag{11}$$

4. The evolution of the plastic strain during yielding is governed by a **flow rule**, formulated in terms of the plastic strain rate, γ:

$$\dot{\varepsilon}^p = \gamma \cdot \text{sign}\left(\sigma - \sigma^{kin}\right) \tag{12}$$

The variable γ plays an important role in this plasticity model. It represents the change in the plastic strain per unit of pseudo-time clocked by the steps of the nonlinear analysis algorithm. We can denote that pseudo-time by t and one analysis increment by Δt. However, we later avoid working with the time variable by defining $\Delta\gamma = \Delta t \cdot \gamma$, thereby shifting focus to the change in plastic strain in each load-step, i.e., $\Delta\gamma$. The key objective in a load-step with yielding is to determine $\Delta\gamma$.

5. The rate of change of the back-stress, i.e., the *kinematic hardening* variable, σ_b, is assumed equal to the rate of change in the plastic strain, multiplied by a material constant, H:

$$\dot{\sigma}_b = \gamma \cdot H \cdot \text{sign}\left(\sigma - \sigma^{kin}\right) \tag{13}$$

6. Several options are available as as *isotropic hardening* rules. A "saturation" type hardening rule is formulated in terms of the material constant δ and the asymptotic yield stress $f_{y\infty}$:

$$f^{iso}(\alpha) = \left(f_{y\infty} - f_y\right)\left(1 - e^{-\delta \cdot \alpha}\right) \tag{14}$$

A simpler and linear isotropic hardening rule features one constant and reads

$$f^{iso}(\alpha) = K \cdot \alpha \tag{15}$$

where this α should not be confused with the α employed in Figure 1. For both isotropic hardening rules, the rate of change of α is assumed equal to the rate of change in the plastic strain:

$$\dot{\alpha} = \gamma \tag{16}$$

KKT and Consistency

The interplay between the yield function, f, and the rate of plastic strain, γ, is at the crux of plasticity theory. When an initially elastic stress state, characterized by $f<0$, reaches $f=0$ then plastic flow, characterized by $\gamma>0$ ensues. The "KKT conditions" from optimization theory enforces the appropriate constraints on f and γ:

$$\gamma \geq 0 \tag{17}$$

$$f \leq 0 \tag{18}$$

$$\gamma \cdot f = 0 \tag{19}$$

Eqs. (17) and (18) are called necessary conditions and Eq. (19) is referred to as the complementary conditions. The acronym KKT is after Harold W. Kuhn and Albert W. Tucker, who published the conditions in a paper on nonlinear programming at a Berkeley symposium in 1951. William Karush had published the conditions in his 1939 Master's thesis, so the conditions are named Karush-Kuhn-Tucker, KKT. In plasticity theory the additional "consistency criterion" is formulated as

$$\gamma \cdot \dot{f} = 0 \tag{20}$$

which says that the value of f cannot change during yielding. In other words, there is either a rate of plastic flow, or there is a rate of change of f in the elastic domain. There are situations where the complementarity condition, $f = 0$, is sufficient, but the consistency criterion, $\dot{f} = 0$, often serves an important role in the determination of plastic flow, particularly with multi-dimensional stress.

Trial Elastic State
The material model presented here is a "total strain" model. This means that the total strain, ε_{n+1} is given to the material in the state determination, not the incremental strain. The objective is to determine the corresponding stress and tangent stiffness. Three history variables are stored from the previously committed state: ε_n^p, σ_{bn}, and α_n. The algorithm proceeds to determine the trial stress (not the Newton-Raphson trial, but a trial within the material) by first assuming an elastic step:

$$\sigma_{n+1}^{trial} = E \cdot \left(\varepsilon_{n+1} - \varepsilon_n^p \right) \tag{21}$$

With the trial stress in Eq. (21), the yield function is evaluated:

$$f_{n+1}^{trial} = \left| \sigma_{n+1}^{trial} - \sigma_n^{kin} \right| - \left(f_y + f_{iso}(\alpha_n) \right) \tag{22}$$

If $f_{n+1}^{trial} < 0$ then the step is elastic and the trial state is the true stage. If $f_{n+1}^{trial} > 0$ then yielding takes place and the plastic strain needs to be determined. That is the subject of the following subsections.

Time Integration
Eqs. (12), (13), and (16) are first-order differential equations that govern the evolution of plastic strain, back-stress, and isotropic hardening. Numerical methods to solve such differential equations include Euler's forward and backward methods and the midpoint rule. To understand those methods, consider Eqs. (12), (13), and (16) written in the generic format

$$\dot{x}(t) = r\left(x, t \right) \tag{23}$$

where t is time and a single dot implies one derivative with respect to time. A class of solution methods is expressed by

$$\begin{aligned}
x(t + \Delta t) &= x(t) + \Delta t \cdot r\left((1 - \theta) \cdot x(t) + \theta \cdot x(t + \Delta t) \right) \\
&= x(t) + \Delta t \cdot r\left(x(t) + \theta \cdot \left(x(t + \Delta t) - x(t) \right) \right)
\end{aligned} \tag{24}$$

where $\theta \in [0,1]$ determines where in the interval from t to $t+\Delta t$ the right-hand side, r, is evaluated. With the notation $t=t_n$, $t_{n+1}=t+\Delta t$, $t_{n+\theta}=t+\theta \cdot \Delta t$, $x(t)=x_n$, $x(t+\Delta t)=x_{n+1}$, and $x(t+\theta \cdot \Delta t)=x_{n+\theta}$, Eq. (24) takes the form

$$x_{n+1} = x_n + \Delta t \cdot r\left(t_{n+\theta}, x_{n+\theta}\right) \tag{25}$$

Common terminology is that $\theta=0$ means the forward Euler method (explicit), $\theta=1$ means the backward Euler method (implicit), and $\theta=1/2$ means the midpoint rule. Addressing Eqs. (12), (13), and (16), those methods are applied to the evolution of e^p, σ^{kin}, and α during plastic flow:

$$\varepsilon_{n+1}^p = \varepsilon_n^p + \underbrace{\Delta t \cdot \gamma}_{\equiv \Delta \gamma} \cdot \text{sign}(\sigma_{n+\theta} - \sigma_{n+\theta}^{kin}) \tag{26}$$

$$\alpha_{n+\theta} = \alpha_n + \Delta \gamma \tag{27}$$

$$\sigma_{n+1}^{kin} = \sigma_n^{kin} + \Delta \gamma \cdot \text{sign}\left(\sigma_{n+\theta} - \sigma_{n+\theta}^{kin}\right) \tag{28}$$

where the definition $\Delta \gamma = \Delta t \cdot \gamma$ is made and employed.

Return Mapping
After finding that $f_{n+1}^{trial} > 0$ a "return mapping" algorithm corrects for the plastic flow that is occurring, bringing the yield function back to zero. The increment in plastic strain, $\Delta \gamma$, is determined from the complementarity condition, $f = 0$, and the consistency criterion, $\dot{f} = 0$. Applying the previously adopted notation, the yield function reads

$$f_{n+\theta}^{trial} = \left|\sigma_{n+\theta} - \sigma_{n+\theta}^{kin}\right| - \left(f_y + f^{iso}(\alpha_{n+\theta})\right) \tag{29}$$

For now, forward Euler, i.e., $\theta=0$, is adopted in the following. That means the ingredients of the yield function are, starting with the stress:

$$\begin{aligned}\sigma_{n+\theta} &= E \cdot \left(\varepsilon_{n+1} - \varepsilon_{n+\theta}^p\right) \\ &= E \cdot \left(\varepsilon_{n+1} - \varepsilon_n^p - \Delta \gamma \cdot \text{sign}\left(\sigma_{n+1}^{trial} - \sigma_n^{kin}\right)\right) \\ &= \sigma_{n+1}^{trial} - E \cdot \Delta \gamma \cdot \text{sign}\left(\sigma_{n+1}^{trial} - \sigma_n^{kin}\right)\end{aligned} \tag{30}$$

where the trial elastic stress is used for now in the signum function, for simplicity, to obtain a linear equation for $\Delta \gamma$ in certain cases considered below. Similarly, the back-stress is

$$\sigma_{n+\theta}^{kin} = \sigma_n^{kin} + \Delta \gamma \cdot H \cdot \text{sign}\left(\sigma_{n+1}^{trial} - \sigma_n^{kin}\right) \tag{31}$$

and the isotropic hardening variable is

$$\alpha_{n+\theta} = \alpha_n + \Delta \gamma \tag{32}$$

Using the KKT complementary condition, $f=0$, to determine the plastic strain increment, $\Delta \gamma$, a few simple cases are first considered. For the case with *no hardening*:

$$f_{n+1}^{trial} = \left| \sigma_{n+1}^{trial} \right| - E \cdot \Delta\gamma - f_y = 0 \quad \Rightarrow \quad \Delta\gamma = \frac{\left| \sigma_{n+1}^{trial} \right| - f_y}{E} \tag{33}$$

For the case of *linear isotropic* hardening:

$$f_{n+1}^{trial} = \left| \sigma_{n+1}^{trial} \right| - E \cdot \Delta\gamma - f_y - K \cdot \alpha_n - K \cdot \Delta\gamma = 0 \quad \Rightarrow \quad \Delta\gamma = \frac{\left| \sigma_{n+1}^{trial} \right| - f_y - K \cdot \alpha_n}{E + K} \tag{34}$$

For the case of *kinematic* hardening and *linear isotropic* hardening:

$$f_{n+1}^{trial} = \left| \sigma_{n+1}^{trial} - \sigma_n^{kin} \right| - H \cdot \Delta\gamma - E \cdot \Delta\gamma - f_y - K \cdot \alpha_n - K \cdot \Delta\gamma = 0$$

$$\Downarrow \tag{35}$$

$$\Delta\gamma = \frac{\left| \sigma_{n+1}^{trial} - \sigma_n^{kin} \right| - f_y - K \cdot \alpha_n}{E + H + K}$$

Adding *saturation isotropic* hardening, the yield function is nonlinear in terms of $\Delta\gamma$:

$$\begin{aligned} f_{n+1}^{trial} = &\left| \sigma_{n+1}^{trial} - \sigma_n^{kin} \right| - H \cdot \Delta\gamma - E \cdot \Delta\gamma - \\ &f_y - K \cdot \alpha_n - K \cdot \Delta\gamma - \left(f_{y\infty} - f_y \right)\left(1 - e^{-\delta \cdot (\alpha_n + \Delta\gamma)} \right) \\ &= 0 \end{aligned} \tag{36}$$

Because it is nonlinear in $\Delta\gamma$ it is solved by Newton's algorithm. That would have been necessary also with the other cases unless forward Euler was used with the trial stress employed in the signum function. The required derivative needed in the Newton algorithm is

$$\frac{\partial f_{n+1}}{\partial \Delta\gamma} = -H - E - K - \delta \cdot \left(f_{y\infty} - f_y \right) \cdot e^{-\delta \cdot (\alpha_n + \Delta\gamma)} \tag{37}$$

Consistent Tangent Stiffness

For nonlinear analysis with the Newton-Raphson algorithm it is necessary to return the tangent stiffness alongside the stress. It can be determined by the consistency criterion, but the complementarity condition is first employed here. If the step is elastic, the stiffness is

$$\frac{\partial \sigma_{n+1}}{\partial \varepsilon_{n+1}} = E \tag{38}$$

If the step is plastic, the stiffness is, according to Eq. (30)

$$\begin{aligned} \frac{\partial \sigma_{n+1}}{\partial \varepsilon_{n+1}} &= \frac{\partial \sigma_{n+1}^{trial}}{\partial \varepsilon_{n+1}} - E \cdot \frac{\partial \Delta\gamma}{\partial \varepsilon_{n+1}} \\ &= E - E \cdot \frac{\partial \Delta\gamma}{\partial \varepsilon_{n+1}} \end{aligned} \tag{39}$$

To determine the derivative of $\Delta\gamma$ we differentiate the yield function, $f=0$, with respect to the strain. For the case with *no hardening*, given in Eq. (33), that gives

$$E - E \cdot \frac{\partial \Delta \gamma}{\partial \varepsilon_{n+1}} = 0 \quad \Rightarrow \quad \frac{\partial \Delta \gamma}{\partial \varepsilon_{n+1}} = 1 \tag{40}$$

which substituted into Eq. (39) naturally gives, for a perfectly plastic material:

$$\frac{\partial \sigma_{n+1}}{\partial \varepsilon_{n+1}} = 0 \tag{41}$$

For the case with only *linear isotropic* hardening, given in Eq. (34):

$$E - E \cdot \frac{\partial \Delta \gamma}{\partial \varepsilon_{n+1}} - f_y - K \cdot \alpha_n - K \cdot \frac{\partial \Delta \gamma}{\partial \varepsilon_{n+1}} = 0 \quad \Rightarrow \quad \frac{\partial \Delta \gamma}{\partial \varepsilon_{n+1}} = \frac{E}{E+K} \tag{42}$$

which substituted into Eq. (39) gives

$$\frac{\partial \sigma_{n+1}}{\partial \varepsilon_{n+1}} = E - \frac{E^2}{E+K} = \frac{E \cdot K}{E+K} \tag{43}$$

Adding *kinematic* hardening, as was done in Eq. (35):

$$E - H \cdot \frac{\partial \Delta \gamma}{\partial \varepsilon_{n+1}} - E \cdot \frac{\partial \Delta \gamma}{\partial \varepsilon_{n+1}} - K \cdot \frac{\partial \Delta \gamma}{\partial \varepsilon_{n+1}} = 0$$
$$\Downarrow \tag{44}$$
$$\frac{\partial \Delta \gamma}{\partial \varepsilon_{n+1}} = \frac{E}{E+H+K}$$

which substituted into Eq. (39) gives

$$\frac{\partial \sigma_{n+1}}{\partial \varepsilon_{n+1}} = \frac{E \cdot (H+K)}{E+H+K} \tag{45}$$

For the case with only *saturation isotropic* hardening, given in Eq. (36):

$$E - E \cdot \frac{\partial \Delta \gamma}{\partial \varepsilon_{n+1}} - \delta \cdot \left(f_{y\infty} - f_y \right) \cdot e^{-\delta \cdot (\alpha_n + \Delta \gamma)} \cdot \frac{\partial \Delta \gamma}{\partial \varepsilon_{n+1}} = 0$$
$$\Downarrow \tag{46}$$
$$\frac{\partial \Delta \gamma}{\partial \varepsilon_{n+1}} = \frac{E}{E + \delta \cdot \left(f_{y\infty} - f_y \right) \cdot e^{-\delta \cdot (\alpha_n + \Delta \gamma)}}$$

which substituted into Eq. (39) gives

$$\frac{\partial \sigma_{n+1}}{\partial \varepsilon_{n+1}} = E - \frac{E^2}{E + \delta \cdot \left(f_{y\infty} - f_y \right) \cdot e^{-\delta \cdot (\alpha_n + \Delta \gamma)}} \tag{47}$$

The consistency criterion, i.e., $\dot{f} = 0$, is now employed to determine the tangent stiffness. As an example, consider the yield function

$$f = \left| \sigma - \sigma^{kin} \right| - \left(f_y + K \cdot \alpha \right) \tag{48}$$

The time-derivative employed in the consistency criterion is

$$
\begin{aligned}
\dot{f} \quad &= \quad & \frac{\partial f}{\partial \sigma} \cdot \dot{\sigma} \quad &+\quad & \frac{\partial f}{\partial \sigma^{kin}} \cdot \dot{\sigma}^{kin} \quad &+\quad & \frac{\partial f}{\partial \alpha} \cdot \dot{\alpha} \\
&= \quad & \text{sign}\left(\sigma - \sigma^{kin}\right) \cdot \dot{\sigma} \quad &-\quad & \text{sign}\left(\sigma - \sigma^{kin}\right) \cdot \dot{\sigma}^{kin} \quad &-\quad & K \cdot \dot{\alpha} \\
&= \quad & \text{sign}\left(\sigma - \sigma^{kin}\right) \cdot E \cdot \dot{\varepsilon}^{e} \quad &-\quad & H \cdot \gamma \quad &-\quad & K \cdot \gamma \\
&= \quad & \text{sign}\left(\sigma - \sigma^{kin}\right) \cdot E \cdot \left(\dot{\varepsilon} - \dot{\varepsilon}^{p}\right) \quad &-\quad & H \cdot \gamma \quad &-\quad & K \cdot \gamma \\
&= \quad & \text{sign}\left(\sigma - \sigma^{kin}\right) \cdot E \cdot \dot{\varepsilon} - E \cdot \gamma \quad &-\quad & H \cdot \gamma \quad &-\quad & K \cdot \gamma
\end{aligned}
\tag{49}
$$

Setting $\dot{f} = 0$ and solving for the plastic strain rate yields

$$
\gamma = \frac{\text{sign}\left(\sigma - \sigma^{kin}\right) \cdot E}{E + H + K} \cdot \dot{\varepsilon}
\tag{50}
$$

Now consider the stress-strain relationship written on rate form:

$$
\dot{\sigma} = E \cdot \dot{\varepsilon}^{e} = E \cdot \left(\dot{\varepsilon} - \dot{\varepsilon}^{p}\right)
\tag{51}
$$

and substitute the plastic strain in Eq. (12) followed by the strain rate in Eq. (50):

$$
\begin{aligned}
\dot{\sigma} &= E \cdot \dot{\varepsilon}^{e} \\
&= E \cdot \left(\dot{\varepsilon} - \frac{E}{E + H + K} \cdot \dot{\varepsilon} \right) \\
&= \left(\frac{E \cdot (H + K)}{E + H + K} \right) \cdot \dot{\varepsilon}
\end{aligned}
\tag{52}
$$

where the tangent stiffness in the last parenthesis matches Eq. (45). Notice that the inclusion of saturation isotropic hardening will result in the tangent stiffness

$$
\frac{\partial \sigma_{n+1}}{\partial \varepsilon_{n+1}} = \frac{E \cdot \left(H + K + \left[\delta \cdot \left(f_{y\infty} - f_{y} \right) \cdot e^{-\delta \cdot (\alpha_{n} + \Delta \gamma)} \right] \right)}{E + \left(H + K + \left[\delta \cdot \left(f_{y\infty} - f_{y} \right) \cdot e^{-\delta \cdot (\alpha_{n} + \Delta \gamma)} \right] \right)}
\tag{53}
$$

Python
The material model with the hardening models described above is implemented in Python and posted on this website under the name G2 Uniaxial Plasticity Class.

References

I particularly recommend two books on plasticity theory (Lubliner 2008; Simo and Hughes 1998).

Lubliner, J. (2008). *Plasticity Theory*. Dover.

Simo, J. C., and Hughes, T. J. R. (1998). *Computational Inelasticity*. Springer-Verlag New York.