Tapered Beams

Most beams considered on this website are prismatic. That means the cross-section does not change along the length of the beam. What is unique about tapered beams, such as those shown in Figure 1, is primarily the distribution of shear stress over the cross-section, and the presence of vertical axial stress. That is addressed shortly. However, it is first highlighted that deformations and stiffness coefficients for tapered beams are obtained in a straightforward manner, using the principle of virtual forces, in the same manner as for other beam cases.

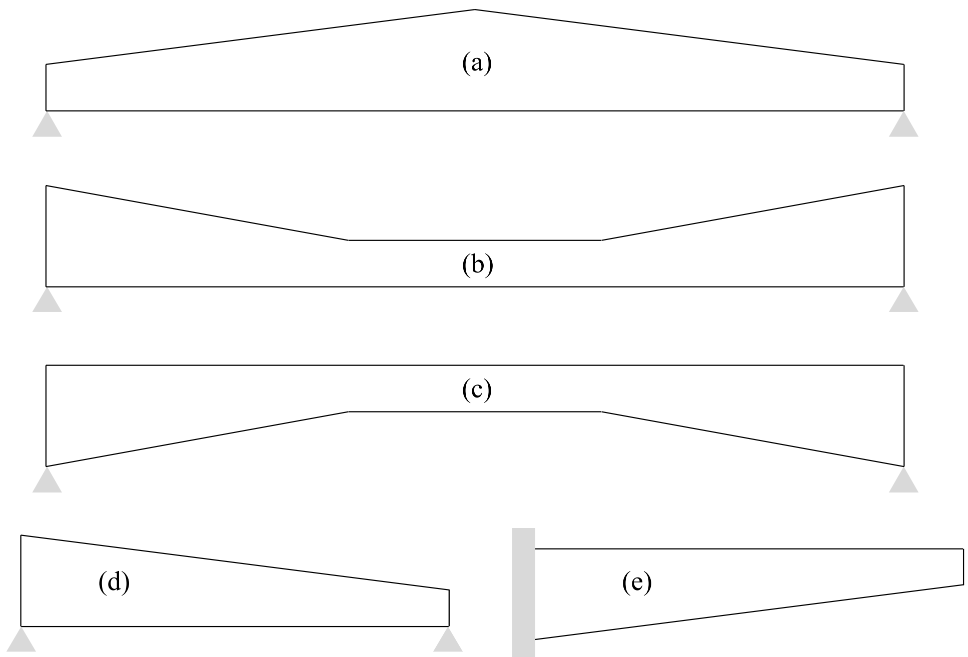

Figure 1: Examples of tapered beams.

Deformations and Stiffness Matrix

Consider the single-tapered beam in Figure 1d) as a beam in its Basic configuration, i.e., with one rotational degree of freedom at each end. Let the cross-section width be constant and equal to b, and let the cross-section height vary linearly from h_1 to h_2. The linear variation of the cross-section height means the moment of inertia exhibits a cubic variation along the beam axis:

$$I = \frac{b \cdot h^3}{12} = \frac{b \cdot \left(h_1 + \frac{h_2 - h_1}{L} \cdot x \right)^3}{12} \qquad (1)$$

The calculation of the end rotation caused by a unit moment at the same location, or at the other end, is an example of deformation calculations. In turn, that will also provide the stiffness matrix. The vehicle for calculating the deformations is the principle of virtual forces:

$$\delta F \cdot \Delta = \int_0^L \delta M \cdot \frac{M}{EI}\, dx + \int_0^L \delta V \cdot \frac{V}{GA_v}\, dx \tag{2}$$

where symbols are defined in the document on the "unit virtual load method." In this case, the left-hand side should be interpreted as a unit virtual moment multiplied by the real rotation that is sought. The second term in Eq. (2) represents shear deformation, with $G=E/(2(1+\nu))$ and $A_v=(5/6)bh$. Figure 2 shows the real and virtual bending moment diagrams that enter into the calculation of rotations. The ultimate objective is to establish the flexibility matrix, followed by inversion to obtain the stiffness matrix. A flexibility coefficient, f_{ij}, is the deformation at degree of freedom number i due to a unit force along degree of freedom number j. For that reason, the flexibility coefficient f_{11} is, with moment diagrams taken from Figure 2:

$$f_{11} = \int_0^L \left(1-\frac{x}{L}\right) \cdot \frac{\left(1-\frac{x}{L}\right)}{E \cdot \frac{b \cdot \left(h_1 + \frac{h_2-h_1}{L} \cdot x\right)^3}{12}}\, dx \tag{3}$$

The other flexibility coefficients are calculated in a similar manner, observing that $f_{12}=f_{21}$. However, the resulting expressions are somewhat intricate and simply evaluated in a single-tapered beam example posted near this document.

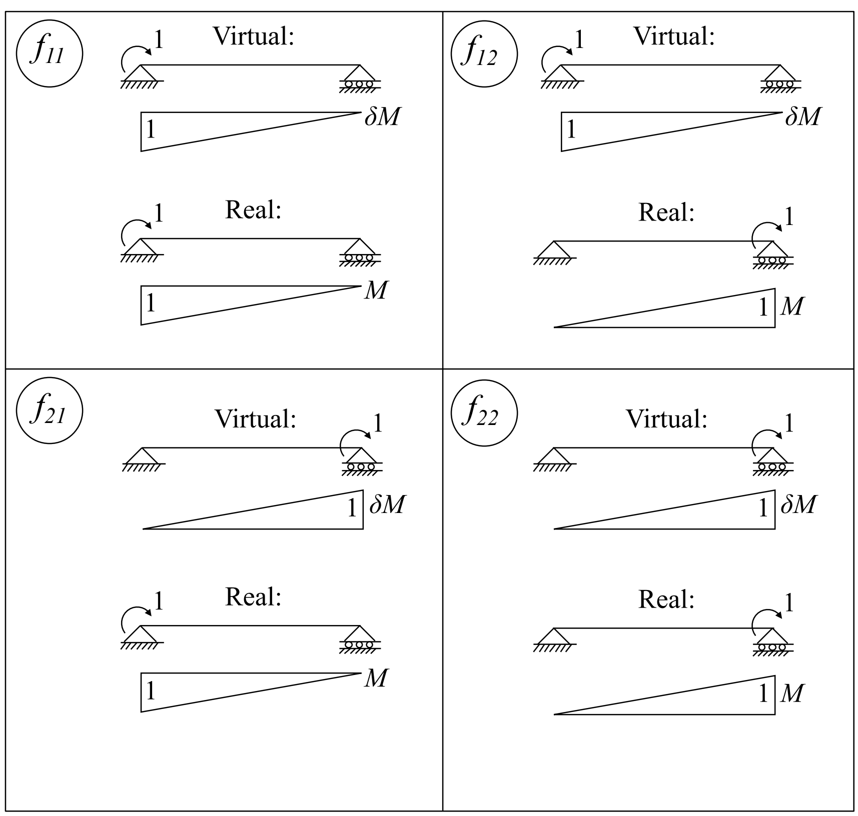

Figure 2: Basic beam cases for virtual work computations.

Once the flexibility matrix, $\mathbf{f}_b$, in the Basic element configuration is established, the corresponding stiffness matrix, $\mathbf{K}_b$, is obtained simply by inversion: $\mathbf{K}_b = \mathbf{f}_b^{-1}$.

Shear Stress

It is in the calculation of stresses that tapered beams are conceptually different from prismatic beams. Specifically, it is necessary to revisit the equilibrium equations that give shear flow and shear stress in the cross-section. It is also necessary to recognize vertical axial stresses in the cross-section. One reference for this work is the Strength of Materials book by Timoshenko (1956). Another reference, addressing a slightly different case, but with more details, is a research paper based on work by Norris and published by the U.S. Forest Service (1965).

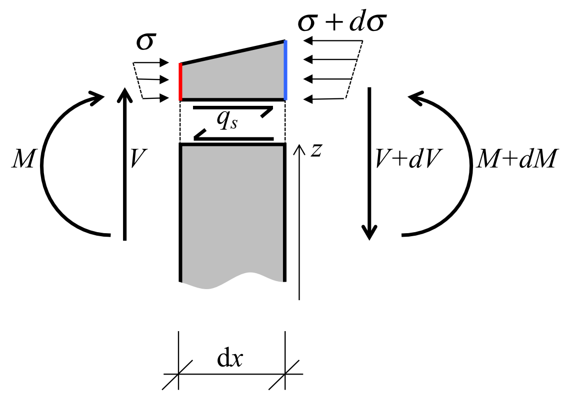

Figure 3: Revised equilibrium for shear flow.

To get started, a revised version of a figure that appears in the Euler-Bernoulli beam document posted on this website is provided in Figure 3. It visualizes how the upper edge of an infinitesimal portion of a tapered beam changes height between the location x and $x+dx$. However, in order to quantify that change in height relative to the change in the full cross-section height, a distinction is made, following the previously mentioned work by Timoshenko and Norris. Observe in Figure 4 that Timoshenko considered a double-tapered beam while Norris et al. considered a single-tapered beam. As a result, Timoshenko let the z-axis run from the centroid of the cross-section while Norris let it run from the horizontal plane of the beam, regardless of whether that plane is at the top or at the bottom. Importantly, the z-axis is perpendicular to a horizontal axis in both cases.

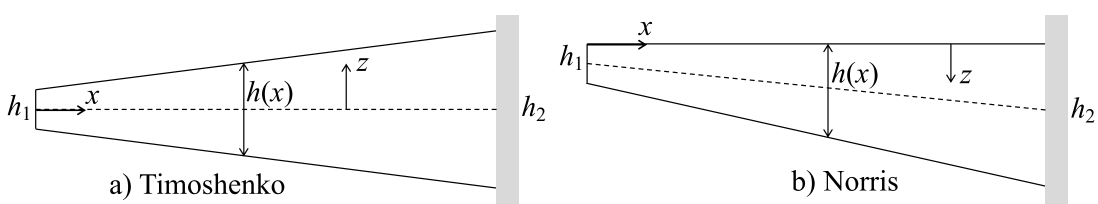

Figure 4: Two distinct cases; notice horizontal upper edge in b).

Before returning to Figure 3 to integrate axial stress one might question the validity of the basic axial stress formula $\sigma = M/I \cdot z$. Timoshenko (1956) addresses that point by comparing axial stress values from exact formulas for different degrees of tapering.

Unless the tapering is very large, the basic axial stress formula is acceptable. Only for a double-tapered case with a 40° angle between the inclined surfaces does the error reach 10%. That means the axial stress for the *z*-definition in Figure 4a) is

$$\sigma = \frac{M}{I} \cdot z = \frac{12 \cdot M}{b \cdot h^3} \cdot z \tag{4}$$

and the axial stress for the z-definition in Figure 4b) is

$$\sigma = \frac{M}{I} \cdot \left(z - \frac{h}{2}\right) = \frac{12 \cdot M}{b \cdot h^3} \cdot \left(z - \frac{h}{2}\right) \tag{5}$$

where both formulas have introduced the assumption of a rectangular cross-section with width *b* and total height *h*. Now consider horizontal equilibrium of the “cut” piece of the beam segment in Figure 3, as is customary to determine shear flow. Because of the different axis definitions used by Timoshenko and Norris, the following equilibrium equations emerge, adopting colours that match those used for the respective surfaces in Figure 3:

$$q_s \cdot dx = b \cdot \int_z^{h+dh} \frac{12 \cdot (M + dM)}{b \cdot (h + dh)^3} \cdot \left(z - \frac{h + dh}{2}\right) dz - b \cdot \int_z^{h} \frac{12 \cdot M}{b \cdot h^3} \cdot \left(z - \frac{h}{2}\right) dz \tag{6}$$

Cancelling *b* and evaluating the integrals yields

$$q_s \cdot dx = \frac{6 \cdot (M + dM)}{(h + dh)^3} \cdot \left((h + dh) \cdot z - z^2\right) - \frac{6 \cdot M}{h^3} \cdot (hz - z^2) \tag{7}$$

Differentiating through by *dx*, the US Forest Service (1965) describes a series expansion, neglecting products of differentials, to obtain the following expression for the shear flow for the case in Figure 4b):

$$\boxed{q_s = \frac{6 \cdot M}{h^2}\left(3 \cdot \left(\frac{z}{h}\right)^2 - 2 \cdot \frac{z}{h}\right) \cdot \frac{dh}{dx} + \frac{6}{h}\left(\frac{z}{h} - \left(\frac{z}{h}\right)^2\right) \cdot \frac{dM}{dx}} \tag{8}$$

With reference to Figure 4a), Timoshenko (1956) starts by considering the specific case of a cantilever with vertical point load at the free end where *x*=0 and writes equilibrium as

$$q_s \cdot dx = b \cdot \int_z^{\frac{h+dh}{2}} \frac{12 \cdot P \cdot z}{b} \cdot \left(\frac{x}{h^3} + \frac{d}{dx}\left(\frac{x}{h^3}\right) dx\right) dz - b \cdot \int_z^{\frac{h}{2}} \frac{12 \cdot P \cdot x \cdot z}{b \cdot h^3} dz \tag{9}$$

Cancelling *b* and *dx* and evaluating the integrals yields

$$q_s = \frac{3P \cdot x}{h^2} \cdot \frac{dh}{dx} + 6P\left(\frac{h^2}{4} - z^2\right) \cdot \frac{d}{dx}\left(\frac{x}{h^3}\right) \tag{10}$$

Timoshenko (1956) then generalizes that expression to accommodate any variation in the bending moment, which gives the following formula for the shear flow for the case in Figure 4a):

$$\boxed{q_s = \frac{3M}{h^2} \cdot \frac{dh}{dx} + 6\left(\frac{h^2}{4} - z^2\right) \cdot \frac{d}{dx}\left(\frac{M}{h^3}\right)} \tag{11}$$

Recognize that $V=dM/dx$ and consider first, for reference, a prismatic beam, i.e., $dh/dx=0$. In that case, Eqs. (8) and (11) both give zero shear flow on the upper and lower edges, as boundary conditions on the shear flow dictate. Furthermore, for both formulas, the shear *stress* in the middle of the cross-section is $3V/2A$, as known from basic mechanics. Beyond that, observations are made in a separate example document posted near this one, considering a single-tapered beam. Vertical stresses, given in the U.S. Forest Service paper, are not studied here, for now.

References

Timoshenko, S. (1956). "Strength of Materials." Third Edition, D. van Nostrand Company, Inc.

U.S. Forest Service (1965). "Deflection and Stresses of Tapered Wood Beams." Research Paper.