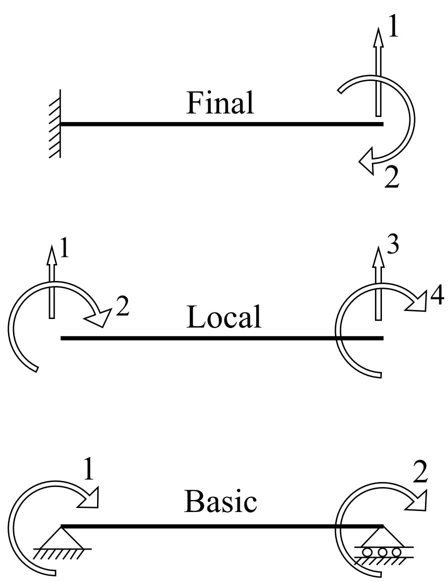
Stiffness vs. Flexibility Matrix

Much of computational structural analysis is centred around the stiffness method and the principle of virtual displacements. That implies that the displacements, including rotations, are the primary unknowns. However, from fundamental structural analysis we know that there are two options: The displacement-based approach and the force-based approach. This document outlines some of the differences between those two approaches in a matrix structural analysis context. In the displacement-based approach, the governing equations for the final structural degrees of freedom are equilibrium equations: $\mathbf{K}_f \, \mathbf{u}_f = \mathbf{F}_f$, where $\mathbf{K}_f$ is the final stiffness matrix. In the force-based approach, the governing equations are compatibility equations: $\mathbf{u}_{f,0} + \mathbf{f}_f \, \mathbf{F}_f =$, where $\mathbf{f}_f$ is the final flexibility matrix and $\mathbf{u}_{f,0}$ is the displacements due to the external loads, temperature changes, settlements, or whatever is acting on the structure. The focus in this document is on the assembly of $\mathbf{K}_f$ and $\mathbf{f}_f$ using transformation matrices. The process is exemplified utilizing the figure shown here:



Stiffness Matrix

This is the "well known" approach from the computational stiffness method document. The Global element configuration is omitted here, because the element orientation is horizontal. Also, the All configuration is omitted for simplicity, going directly from the Local to the final configuration. Letting the element length be L, the transformation from the Basic to the Local configuration reads

$$\text{Tbl} = \left\{\left\{-\frac{1}{L}, 1, \frac{1}{L}, 0\right\}, \left\{-\frac{1}{L}, 0, \frac{1}{L}, 1\right\}\right\};$$

```
Tbl // MatrixForm
```

which yields: $\begin{pmatrix} -\frac{1}{L} & 1 & \frac{1}{L} & 0 \\ -\frac{1}{L} & 0 & \frac{1}{L} & 1 \end{pmatrix}$

The transformation from the Local to the Final configuration is here established, column-by-column, in the manner described in the document on the computational stiffness method:

```
Tlf = {{0, 0}, {0, 0}, {1, 0}, {0, 1}};
Tlf // MatrixForm
```

which yields: $\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$

Because the Basic stiffness matrix is...

$$\text{Kb} = \left\{\left\{\frac{4\,EI}{L}, \frac{2\,EI}{L}\right\}, \left\{\frac{2\,EI}{L}, \frac{4\,EI}{L}\right\}\right\};$$

```
Kb // MatrixForm
```

which yields: $\begin{pmatrix} \frac{4\,EI}{L} & \frac{2\,EI}{L} \\ \frac{2\,EI}{L} & \frac{4\,EI}{L} \end{pmatrix}$

... the final stiffness matrix is:

$$\text{Kf} = \text{Tlf}^{T}.\text{Tbl}^{T}.\text{Kb}.\text{Tbl}.\text{Tlf};$$

```
Kf // MatrixForm
```

which yields: $\begin{pmatrix} \frac{12\,EI}{L^3} & \frac{6\,EI}{L^2} \\ \frac{6\,EI}{L^2} & \frac{4\,EI}{L} \end{pmatrix}$

Using the fact that the flexibility matrix is the inverse of the stiffness matrix, the above derivations leads to the following final flexibility matrix:

```
ff = Inverse[Kf];
ff // MatrixForm
```

which yields: $\begin{pmatrix} \frac{L^3}{3\,EI} & -\frac{L^2}{2\,EI} \\ -\frac{L^2}{2\,EI} & \frac{L}{EI} \end{pmatrix}$

Flexibility Matrix

In the context of matrix structural analysis, the flexibility method is less well known. The starting point here is the Basic flexibility matrix, derived by the principle of virtual forces, containing f_{ij}=displacement along DOF i due to a unit force along DOF j. It reads, for the Basic element configuration:

```
fb = {{L/(3 EI), -L/(6 EI)}, {-L/(6 EI), L/(3 EI)}};
fb // MatrixForm
```

which yields: $\begin{pmatrix} \frac{L}{3\,EI} & -\frac{L}{6\,EI} \\ -\frac{L}{6\,EI} & \frac{L}{3\,EI} \end{pmatrix}$

In order to obtain the Final flexibility matrix, we must omit the Local element configuration. That is because the Local configuration is NOT statically determinate; if we apply a unit force along any given DOF, then the structure will simply accelerate. That is, because we cannot do equilibrium in the unstable Local element configuration, we go directly from the statically determinate Basic configuration to the statically determinate Final configuration. The fundamental idea when establishing the force transformation matrix $\mathbf{T}_{bf,tilde}$ is the same as when we establish the displacement transformation matrix $\mathbf{T}_{bf}$: set degrees of freedom in the configuration "above" equal to one, one at a time. Setting Final DOF number 1 equal to one gives a bending moment diagram with zero at the tip and a value equal to L at the left-hand side base, with tension at the bottom. In order to make the element have the same force pattern in the Basic element configuration, we set the first Basic DOF equal to one. Similarly, the second column of $\mathbf{T}_{bf,tilde}$ is obtained by setting the rotation-DOF in the Final configuration equal to one, implying a constant bending moment diagram with unit value with tension at the top. That force pattern is obtained by setting the first Basic DOF equal to -1 and the second Basic DOF equal to 1:

```
Tbftilde = {{L, -1}, {0, 1}};
Tbftilde // MatrixForm
```

which yields: $\begin{pmatrix} L & -1 \\ 0 & 1 \end{pmatrix}$

That gives the flexibility matrix

```
ff = TbftildeT.fb.Tbftilde;
ff // MatrixForm
```

which yields: $\begin{pmatrix} \frac{L^3}{3\,EI} & -\frac{L^2}{2\,EI} \\ -\frac{L^2}{2\,EI} & \frac{L}{EI} \end{pmatrix}$

That matches what was obtained above with the displacement-based approach followed by the inversion of the stiffness matrix.