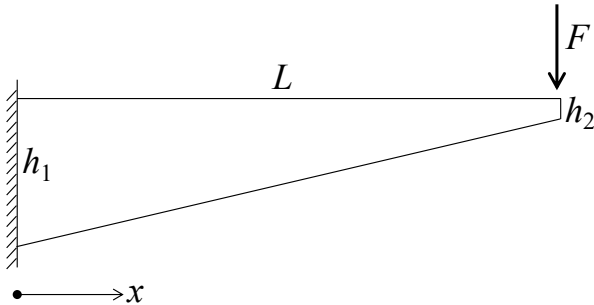# Single-tapered Beam

Consider the cantilevered tapered beam subjected to a point load, *F*, shown in this figure:

The width of the rectangular cross-section is constant and denoted by *b*. The height varies linearly from h_1 to h_2, as shown in the figure. The material is homogeneous and isotropic with modulus of elasticity denoted by *E*. The following input values are used, in N and mm:

```
E = 9500;
L = 6000;
h1 = 1000;
h2 = 200;
b = 200;
F = 5000;
```

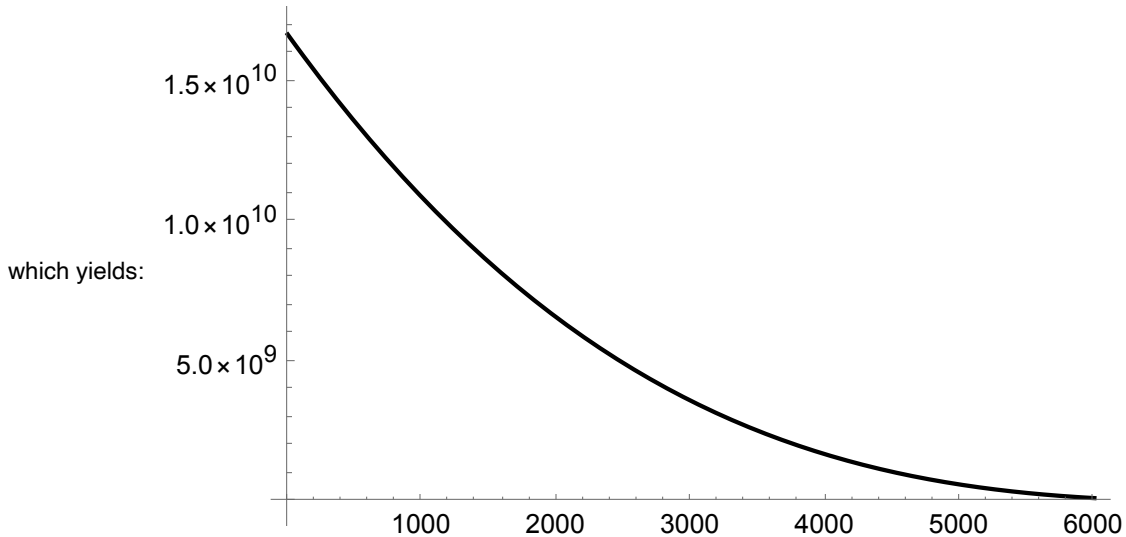The variation in cross-section height is expressed as

```
      h2 – h1
h = h1 + ─────── x;
         L
```

That means the moment of inertia varies as follows along the beam:

```
      b h³
I = ——————— ;
       12
Plot[I, {x, 0, L}, PlotStyle → Black]
```

which yields:

Deformation

The principle of virtual forces is employed in order to determine the tip displacement. However, the "quick integration" formulas cannot be utilized because *EI* varies along the beam. The real bending moment diagram is a triangle that peaks at the value *FL* at the left-hand side end. The virtual moment diagram peaks at the same location with value *L*, because a unit virtual force is placed at the tip. Omitting shear deformation, the deflection is:

$$\int_0^L L \left(1 - \frac{x}{L}\right) \frac{F L \left(1 - \frac{x}{L}\right)}{E I} \, dx \; // \; N$$

which yields: `6.52047`

If shear deformation is included then the deflection is:

```
     E
G = ────── ;
    2 (1 + 0)

     5
Av = ─ b h;
     6
```

$$\int_0^L L \left(1 - \frac{x}{L}\right) \frac{F L \left(1 - \frac{x}{L}\right)}{E I} \, dx + \int_0^L 1 \frac{F}{G \, Av} \, dx \; // \; N$$

which yields: `6.59671`

Another way to approach the problem is to first establish the stiffness matrix for the beam in its Basic element configuration, i.e., with one rotational degree of freedom at each end. The approach is described in the document entitled Tapered Beam, posted near this example, and the result for the present case is the following flexibility matrix:

$$f11 = \int_0^L \left(1 - \frac{x}{L}\right) \frac{\left(1 - \frac{x}{L}\right)}{E I} \, dx;$$

$$f12 = \int_0^L \left(1 - \frac{x}{L}\right) \frac{-\frac{x}{L}}{E I} \, dx;$$

$$f22 = \int_0^L \frac{x}{L} \frac{\frac{x}{L}}{E I} \, dx;$$

```
f = {{f11, f12}, {f12, f22}};
f // N // MatrixForm
```

which yields:
$$\begin{pmatrix} 3.62248 \times 10^{-11} & -5.8512 \times 10^{-11} \\ -5.8512 \times 10^{-11} & 4.15172 \times 10^{-10} \end{pmatrix}$$

The inverse of the flexibility matrix is the stiffness matrix:

```
Kb = Inverse[f];
Kb // N // MatrixForm
```

which yields:
$$\begin{pmatrix} 3.57417 \times 10^{10} & 5.03724 \times 10^9 \\ 5.03724 \times 10^9 & 3.11856 \times 10^9 \end{pmatrix}$$

Transformation to the four-DOF Local element configuration yields:

```
Tbl = {{- 1/L , 1, 1/L , 0}, {- 1/L , 0, 1/L , 1}};
Kl = Tblᵀ.Kb.Tbl;
Kl // N // MatrixForm
```

which yields:
$$\begin{pmatrix} 1359.3 & -6.79649\times10^6 & -1359.3 & -1.3593\times10^6 \\ -6.79649\times10^6 & 3.57417\times10^{10} & 6.79649\times10^6 & 5.03724\times10^9 \\ -1359.3 & 6.79649\times10^6 & 1359.3 & 1.3593\times10^6 \\ -1.3593\times10^6 & 5.03724\times10^9 & 1.3593\times10^6 & 3.11856\times10^9 \end{pmatrix}$$

Removal of the two left-hand side degrees of freedom, and static condensation of the right-hand side rotation gives the vertical stiffness at the tip:

```
Ktip = Kl[[3, 3]] - Kl[[3, 4]]²/Kl[[4, 4]] // N
```

which yields: 766.816

In turn, the vertical displacement with this approach matches the earlier result:

```
F/Ktip
```

which yields: 6.52047

Shear Flow with Timoshenko Tapering (Constant Shear Force)

Consider now the somewhat different, and certainly more convenient, conventions when shear stresses are considered. The figure below shows on the left-hand side the beam considered by Timoshenko in 1956 in his Strength of Materials book. On the right-hand side is the case considered by Norris et al. in the US Forest Service reference mentioned in the document on Tapered Beams posted near this one.
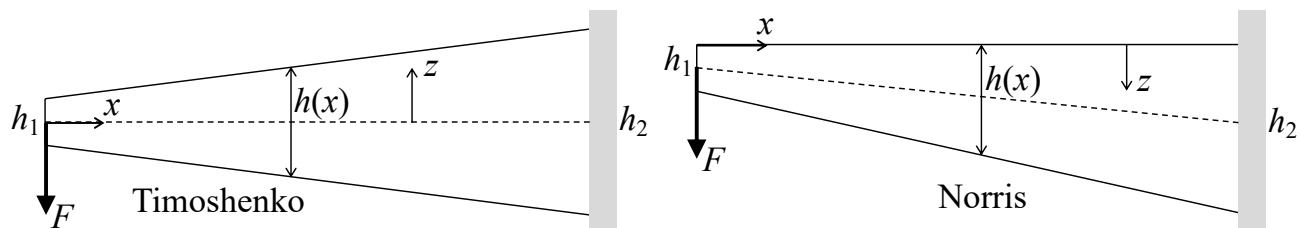
The following input values are used, in units of N and mm:

```
L = 6000;
h1 = 200;
h2 = 400;
F = 5000;
```

The bending moment varies linearly and the shear force is constant along the beam:

```
M = F x;
```

The equation that describes the variation in cross-section height is:

```
h = h1 + (h2 - h1) x ;
                    L
```

Its derivative is needed for the shear flow formulas:

```
dhdx = D[h, x];
```

The line that describes the upper and lower edge is given by:

```
edge = h1 + h2 - h1 x ;
       2     2    L
```

The following derivative is needed in the Timoshenko formula:

```
dMh3dx = D[ M , x];
            h3
```

The Timoshenko formula for the shear flow (shear stress is obtained by dividing by the width, *b*) is:
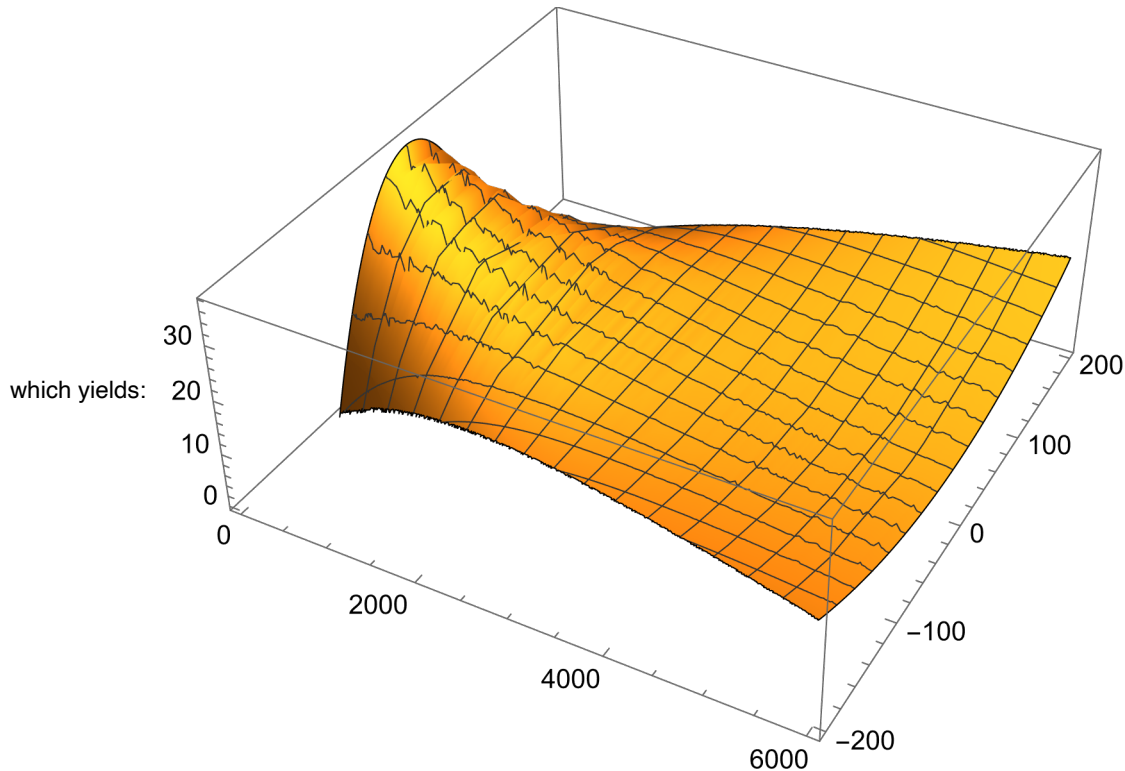
```
qsTimoshenko = 3 M dhdx + 6 ( h2 - z2 ) dMh3dx;
               h2             4
```

The plot of the shear flow is:

```
regionTimoshenko = ImplicitRegion[z < edge && z > -edge && x > 0 && x < L,
    {x, z}];
Plot3D[qsTimoshenko, {x, z} ∈ regionTimoshenko, PlotRange → All]
```

which yields:

Although shear flow and shear stress are not always important considerations in structural engineering, it is remarkable how tapering affects the shear flow distribution over the cross-section. Near the free end, the typical parabola for prismatic beams is recognized. However, it flips to the other side by the time x reaches L. In fact, there is a cross-section somewhere along the beam where the shear flow is entirely constant.

Shear Stress with Norris Tapering (Constant Shear Force)

Using the same input values as above, the following derivative is needed in the Norris formula:
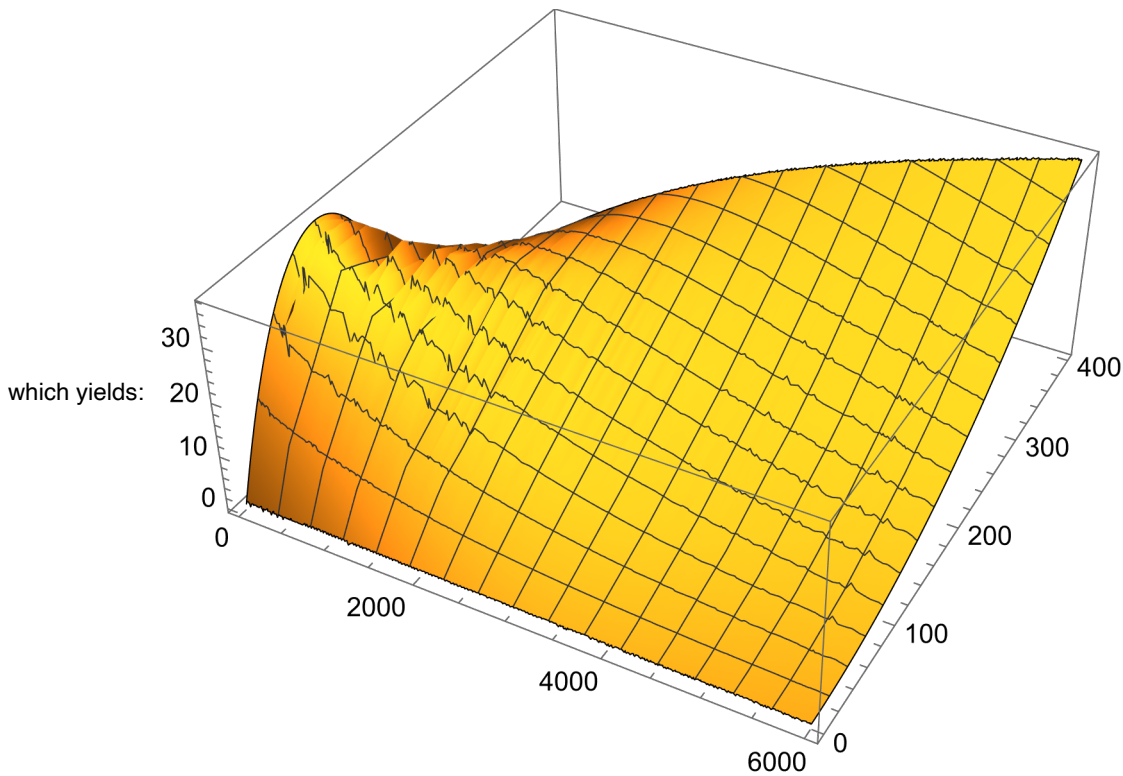
```
dMdx = D[M, x];
```

The Norris formula for the shear flow is:

```
qsNorris = 6 M/h² (3 (z/h)² - 2 z/h) dhdx + 6/h (z/h - (z/h)²) dMdx;
```

The plot of the shear flow is shown below; when interpreting it, remember that the z-axis flows

downwards in the beam considered here, shown in a previous figure:

```
regionNorris = ImplicitRegion[z < h && z > 0 && x > 0 && x < L, {x, z}];
Plot3D[qsNorris, {x, z} ∈ regionNorris, PlotRange → All]
```

which yields:

Again it is observed how tapering significantly affects the shear flow and shear stress. For the kind of tapering here considered by Norris and the US Forest Service reference, there exists a cross-section along which the shear flow varies linearly from zero at the horizontal edge, to some non-zero value at the tapered edge. That is remarkable, but it does not violate proper boundary conditions on zero surface stress. If the axis system was rotated slightly in order to let z be perpendicular to the tapered edge, the shear stress would indeed be zero in that coordinate system

Negative Haunching (Constant Shear Force)

The shear flow results shown above show a significant influence of tapering on the shear flow pattern over the cross-section, compared to prismatic beams. One way to understand the results is to observe that the height of the cross-section increases in the same direction as the moment increases. This means that a potential increase in the amplitude of the axial stresses due to increasing bending is ameliorated by increasing cross-section height. What if the situation was oppositive? That is called negative haunching, illustrated in this figure:
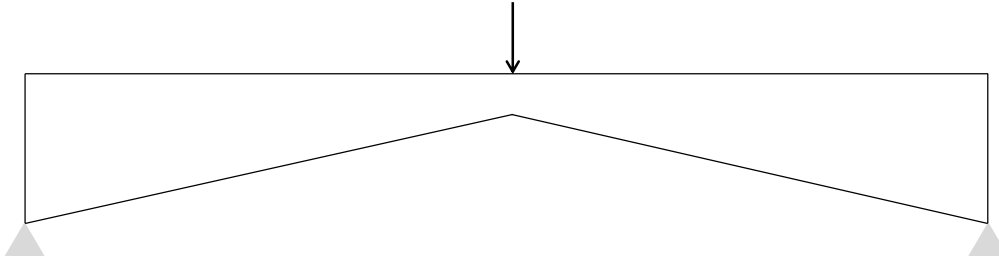
When a point load is placed on the beam, then the bending moment will increase towards the centre of the beam, as long as the loads are downwards-acting. The effect on the shear flow distribution is now investigated simply by switching the values of h_1 and h_2 in the previous Timoshenko example. The input, governing equations, and plot are shown below:

```
L = 6000;
h1 = 400;
h2 = 200;
F = 5000;

M = F x;
```

$$h = h1 + (h2 - h1) \, \frac{x}{L};$$

```
dhdx = D[h, x];
```
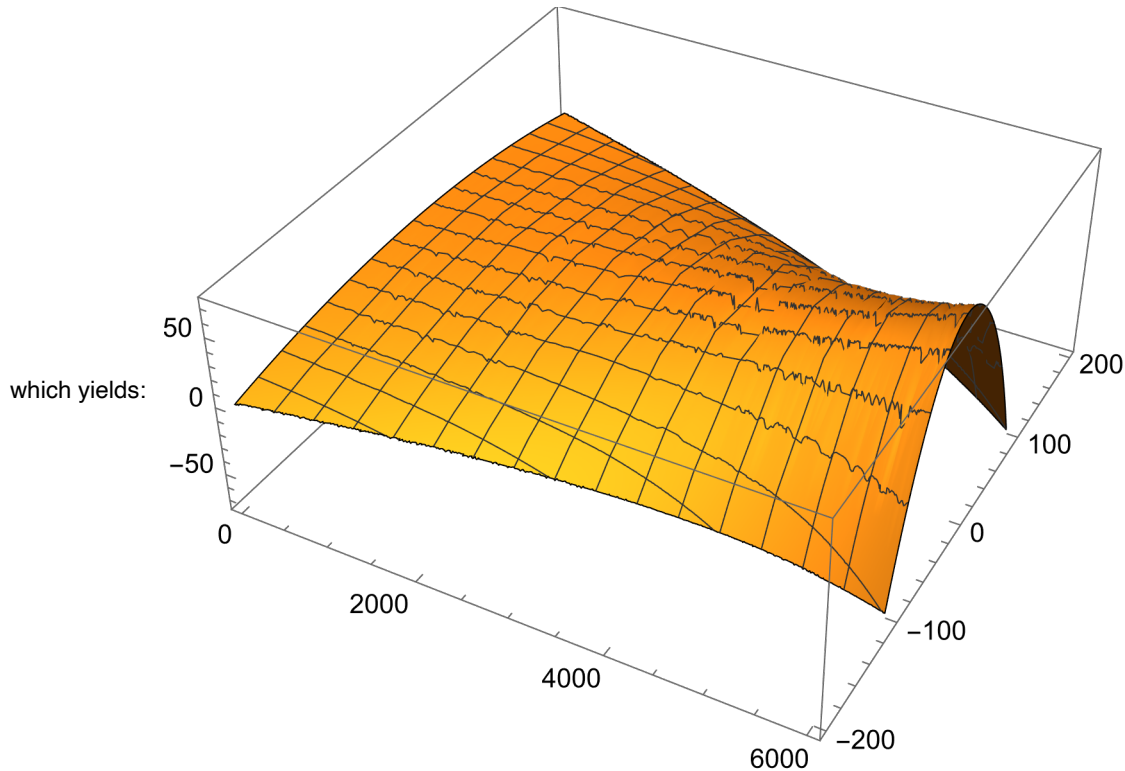
$$edge = \frac{h1}{2} + \frac{h2 - h1}{2} \, \frac{x}{L};$$

$$dMh3dx = D\left[\frac{M}{h^3}, x\right];$$

$$qsTimoshenko = \frac{3\,M}{h^2} \, dhdx + 6\left(\frac{h^2}{4} - z^2\right) dMh3dx;$$

```
regionTimoshenko = ImplicitRegion[z < edge && z > -edge && x > 0 && x < L,
  {x, z}];
Plot3D[qsTimoshenko, {x, z} ∈ regionTimoshenko, PlotRange → All]
```
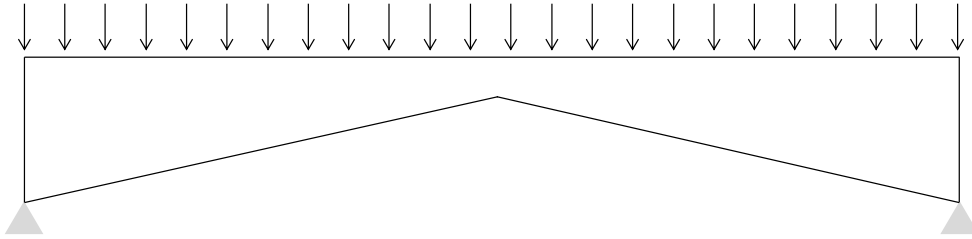
which yields:

The result is as expected; it is still the beam end with the smallest cross-section height that appears to be governing. That is now the right-hand side in the plot, because of the setup of the example, simply switching the values of h_1 and h_2 compared to previous plots. In short, tapering has a significant and interesting effect on the distribution of shear flow, and therefore shear stress, over the cross-section of a tapered beam. However, for constant shear force, it is the beam end with the lowest cross-section height that is governing, regardless of whether the tapering is "positive or negative haunching."

Varying Shear Force

The comment made above about the smallest cross-section being governing, in terms of shear flow in tapered beams, is made for beams with constant shear force. Now consider the beam shown below, where the shear force is zero at the location of the smallest cross-section. From there, the shear force increases linearly towards the supports. In other words, as the cross-section height increases, the shear force also increases. Which cross-section along the beam is now governing?

Consider the same input as above, except now with a uniformly distributed load, *q*, giving a linearly varying shear force:

```
L = 6000;
h1 = 200;
h2 = 400;
q = 1;
```

The maximum moment is, at the location of smallest cross-section:
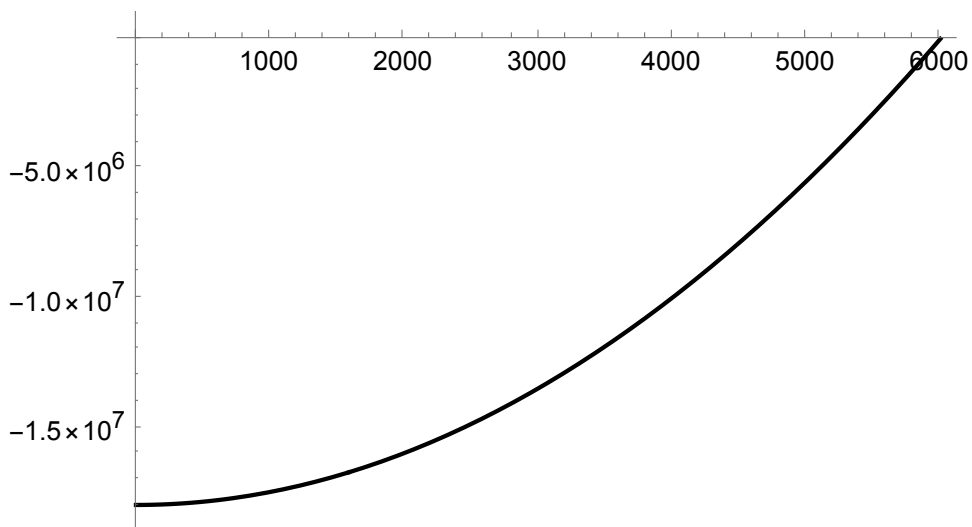
$$\frac{q\ (2\ L)^2}{8} \ // \ N$$

which yields: 1.8×10^7

The bending moment diagram is:

```
M = + q/2 x² - qL²/2 ;
Plot[M, {x, 0, L}, PlotStyle -> Black]
```

which yields:

The shear force diagram is:

```
V = D[M, x];
Plot[V, {x, 0, L}, PlotStyle -> Black]
```
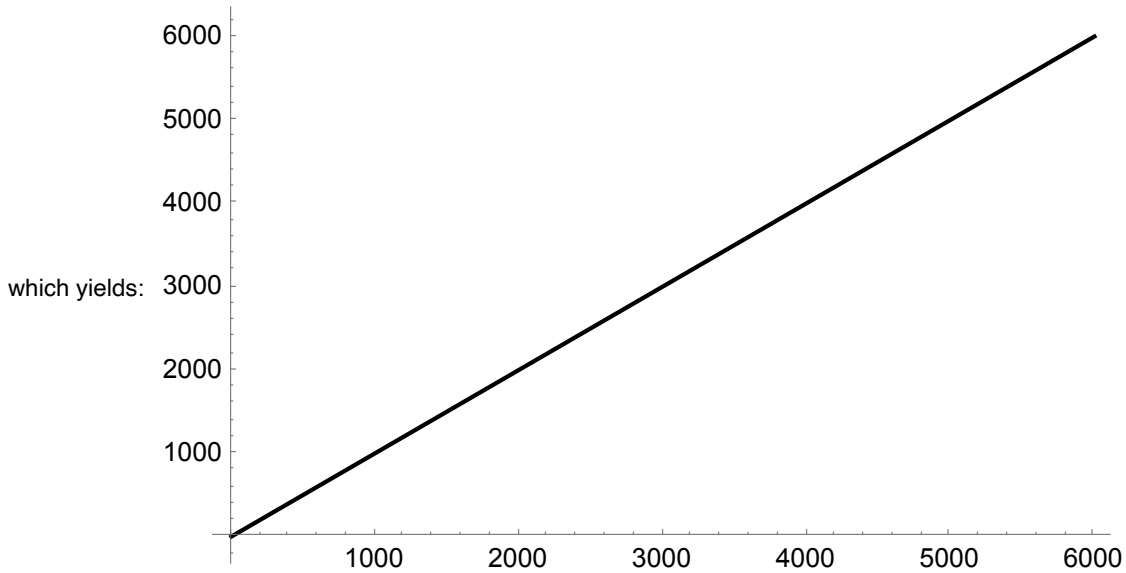
which yields:


```
           x
h = h1 + (h2 - h1) — ;
           L
```

```
dhdx = D[h, x];
```

```
       h1   h2 - h1  x
edge = —— + ——————— — ;
        2      2     L
```

```
          ⎡ M    ⎤
dMh3dx = D ⎢ —— , x⎥ ;
          ⎣ h³   ⎦
```

$$\text{qsTimoshenko} = \frac{3\,M}{h^2}\,\text{dhdx} + 6\left(\frac{h^2}{4} - z^2\right)\text{dMh3dx};$$

```
regionTimoshenko = ImplicitRegion[z < edge && z > -edge && x > 0 && x < L,
    {x, z}];
Plot3D[qsTimoshenko, {x, z} ∈ regionTimoshenko, PlotRange → All]
```

which yields:

The plot now reveals a constant maximum shear flow along the tapered cross-section, i.e., without the smallest cross-section governing.