Plastic Capacity Examples

Examples are provided in this document to demonstrate the hand-calculation of plastic capacities of various problems. Amongst several options, the subscript u for "ultimate" is selected to denote the plastic capacity. That means the ultimate force a structure can carry is F_u, q_u, etc. and the plastic axial and moment capacity of a cross-section is N_u and M_u, respectively. Also note the preface d in the symbol for internal and external work: dW_{int}, dW_{ext}. This is *not* virtual work; rather, it is incremental work carried out after "full" yielding occurs.

Beam with Point Load

Consider the simply supported beam in Figure 1. The objective is to determine the capacity of the beam to carry the point load, F, when the plastic moment capacity of the cross-section, M_u, is assumed to be known. The upper-bound theorem of plastic capacity analysis is suitable for this type of problem. That means we start by assuming the location of plastic hinges; the choice of a midspan hinge is shown by a solid circle in Figure 1.

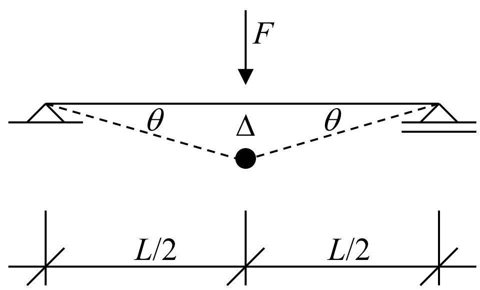

Figure 1: Beam with point load.

Next, the expression for internal incremental plastic work is established, observing that the total rotation at the plastic hinge is 2θ:

$$dW_{int} = 2\theta \cdot M_u \tag{1}$$

The expression for the external incremental plastic work employs the kinematic relationship $\Delta=\theta(L/2)$:

$$dW_{ext} = F \cdot \theta \cdot \frac{L}{2} \tag{2}$$

Setting $dW_{int}=dW_{ext}$ and solving for F yields the plastic capacity of the beam to carry that load:

$$F_u = \frac{4M_u}{L} \tag{3}$$

Beam with Distributed Load

Consider the fixed-fixed beam in Figure 2. Towards the objective of determining the capacity, q_u, of the beam to carry distributed load, the shown mechanism is assumed. It involves three plastic hinges; the rotation is θ for two of them and 2θ for the middle hinge. That means the internal incremental plastic work is

$$dW_{int} = (2\theta + \theta + \theta) \cdot M_u \tag{4}$$

The external work is determined by considering the two sides, on either side of the midspan hinge, separately. That is the reason for the factor 2 in the following formula. On each side, the distributed load is collected into a resultant that conducts work over a displacement that is half of Δ. That means the external work is

$$dW_{ext} = 2 \cdot \left(q \cdot \frac{L}{2}\right)\left(\frac{1}{2} \cdot \frac{L}{2} \cdot \theta\right) \tag{5}$$

Setting dW_{int}=dW_{ext} and solving for q yields the plastic capacity of the beam to carry that load:

$$q_u = \frac{16M_u}{L^2} \tag{6}$$

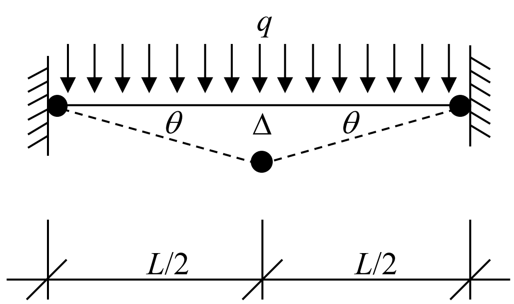

Figure 2: Beam with distributed load.

Frame

Consider the frame in Figure 3, noting that the horizontal beam has twice the plastic cross-section capacity of the columns; i.e., the plastic capacity of the beam is $2M_u$. Adopting the upper-bound theorem, the objective is to determine the capacity of the frame to carry the reference load, F, which acts at two locations, as shown in Figure 3. To that end, the upper-bound theorem of is adopted, followed by the assumption of the deformation mechanism sketched in Figure 3. For this mechanism, the incremental internal plastic work is

$$dW_{int} = (2\theta + \theta + \theta) \cdot M_u + 2\theta \cdot 2M_u \tag{7}$$

The external work done by the two loads is

$$dW_{ext} = F \cdot (0.6L \cdot \theta) + 2F \cdot \left(\frac{L}{2} \cdot \theta\right) \tag{8}$$

Setting dW_{int}=dW_{ext} and solving for F yields the plastic capacity of the frame:

$$F_u = \frac{5M_u}{L} \tag{9}$$

Note that this is the capacity of the frame associated with the specific deformation mechanism sketched in Figure 3. In practical situations, we often need to try a variety of different mechanisms. Because this is an application of the upper-bound theorem, we keep only the lowest capacity value. That value is either correct, or too high if we have not found the correct mechanism. In other words, the value is either correct or unconservative.

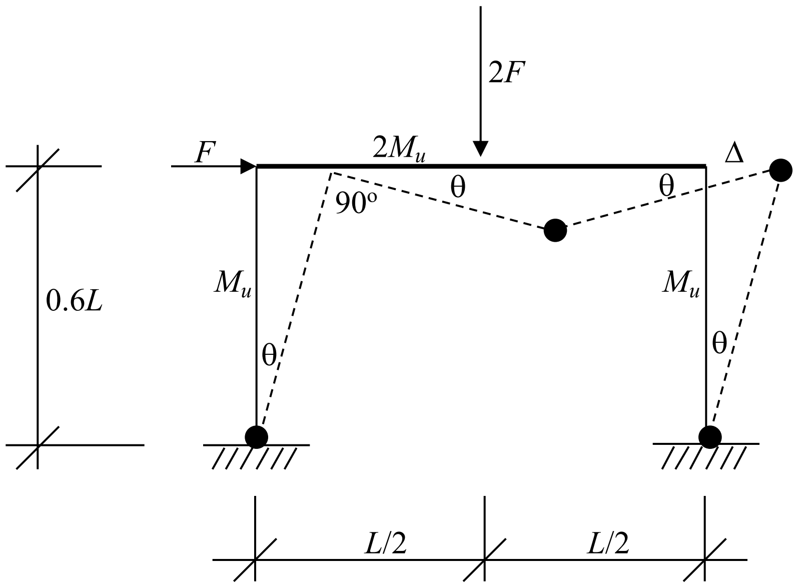

Figure 3: Frame.

Cross-section

Consider Figure 4, showing a rectangular cross-section with two stress blocks. In contrast to the previous examples, the lower-bound of plastic capacity analysis is here applied because that is more convenient at the cross-section level. To that end, the stress in the top half is compression; the stress in the bottom half is tension. Those stress blocks are postulated to take bending moment acting on the cross-section. Simple equilibrium of the resultants of the two stress blocks leads to the following plastic capacity of the cross-section:

$$M_u = \sigma_y \cdot \left(b \cdot \frac{h}{2}\right) \cdot \frac{h}{2} \tag{10}$$

where σ_y is the yield stress of the material.

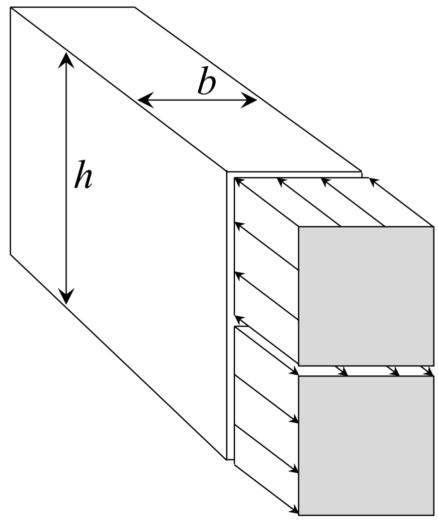

Figure 4: Cross-section.

Isotropic Slab

Consider the slab in Figure 5, which has an isotropic material. In other words, the material properties are the same in all directions. The slab is subjected to a uniformly distributed load, q, and the objective is to determine the plastic capacity value of that load.

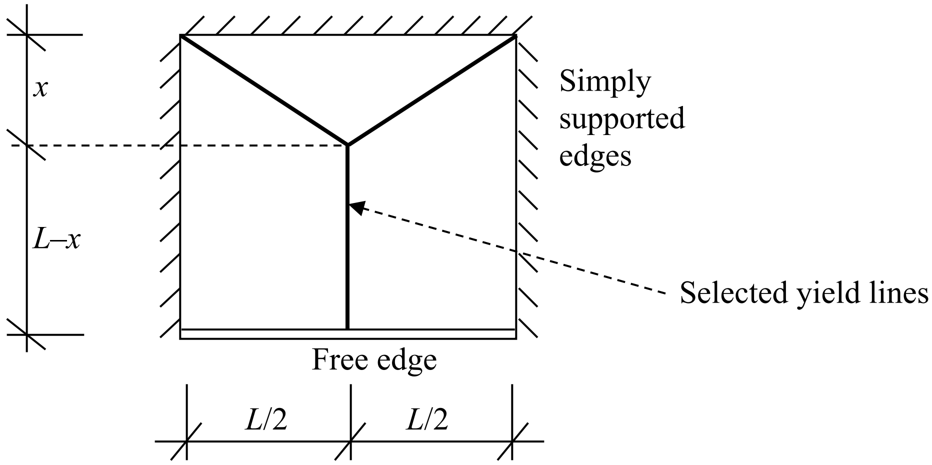

Figure 5: Slab.

Applying the upper-bound theorem, the shown yield lines are assumed. The plastic moment of the cross-section, per unit length, is labelled m and the displacement at the middle of the slab is denoted Δ. At first glance it seems challenging to determine the work done along the inclined yield lines. However, decomposition facilitates the projection of that internal work onto the edges of the plate:

$$dW_{\text{int}} = \sum \vec{M} \cdot \vec{\theta} = \sum_{\substack{\text{each slab} \\ \text{segment}}} ml \cdot \cos(\alpha) \cdot \theta = \underbrace{mL\frac{\Delta}{L/2}}_{\text{Slab segment 1}} + \underbrace{mL\frac{\Delta}{L/2}}_{\text{Slab segment 2}} + \underbrace{mL\frac{\Delta}{x}}_{\text{Slab segment 3}} \tag{11}$$

The external work is determined by first dividing the slab into the three segments shown in Figure 6. Next, the distributed load in each segment is collected into a resultant. The multiplication of the resultant with the displacement at that location yields the external work:

$$dW_{\text{ext}} = \underbrace{q(L-x)\frac{L}{2}\cdot\frac{\Delta}{2} + qx\frac{L}{2}\frac{1}{2}\cdot\frac{\Delta}{3}}_{\text{Slab segment 1}} + \underbrace{q(L-x)\frac{L}{2}\cdot\frac{\Delta}{2} + qx\frac{L}{2}\frac{1}{2}\cdot\frac{\Delta}{3}}_{\text{Slab segment 2}} + \underbrace{qxL\frac{1}{2}\cdot\frac{\Delta}{3}}_{\text{Slab segment 3}} \tag{12}$$

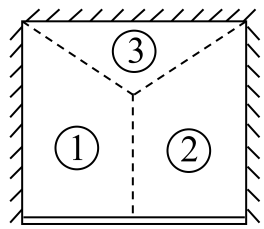

Figure 6: Slab segments.

Setting dW_{int}=dW_{ext} and solving for q yields the plastic capacity:

$$q_u = \frac{\frac{mL}{x} + 4m}{\frac{L^2}{2} - \frac{Lx}{6}} \tag{13}$$

To find the value of x that gives the minimum capacity we set

$$\frac{dq_u}{dx} = 0 \tag{14}$$

and solve, which yields

$$x = L\left(\frac{\sqrt{13}-1}{4}\right) \tag{15}$$

Substituting that value of x into Eq. (13) yields

$$q_u = 14.14 \cdot \frac{m}{L^2} \tag{16}$$

Orthotropic Slab

Consider a generic slab in the x-y plane and let the plastic moment capacity be m_x about the x-axis and m_y about the y-axis. In the following, the notation $m_x = m$ and $m_y = \phi\, m$ is employed. Projection of all rotations onto the x and y axes gives the internal work

$$dW_{\text{int}} = \sum \vec{M}\cdot\vec{\theta} = \sum_{\substack{\text{each slab}\\\text{segment}}} ml_x\cdot\theta_x + \phi\cdot ml_y\cdot\theta_y \tag{17}$$

where l_x, l_y = projection of the length of the yield line onto the x-axis and y-axis, respectively; θ_x = rotation of the slab segment about the x-axis; and θ_y = rotation of the slab segment about the y-axis. Denoting by α the angle between the x-axis and the axis of rotation gives $\theta_x=\theta\cos(\alpha)$ and $\theta_y=\theta\sin(\alpha)$ where θ is the rotation of each slab segment. That gives the following modified expression for the internal work:

$$dW_{\text{int}} = \sum \vec{M} \cdot \vec{\theta} = \sum_{\substack{\text{each slab}\\ \text{segment}}} ml \cdot \cos^2(\alpha) \cdot \theta + \phi \cdot ml \cdot \sin^2(\alpha) \cdot \theta \tag{18}$$

where l is the total yield line length for the segment.