Cylindrical Shells

The most basic case relevant for this document is a cylindrical pressure vessel that we imagine is infinitely long, and uniform in every way in that direction. In addition to constant pressure, those assumptions mean there are no disturbances from the cylinder ends; also, there are no reinforcing "belts" around the cylinder at any location. The problem is visualized in Figure 1. The following notation is employed: p=internal pressure, t=wall thickness, r=inner radius, σ=hoop stress.

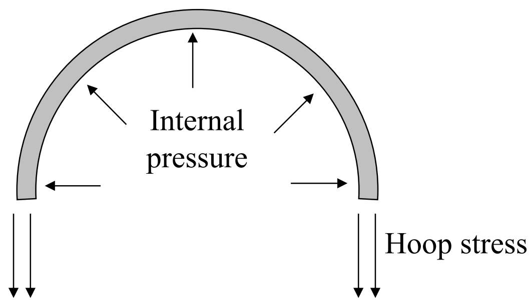

Figure 1: Hoop stress.

With that notation, vertical equilibrium of Figure 1 yields the basic hoop stress formula:

$$p \cdot 2r = \sigma \cdot 2t \Rightarrow \sigma = p \cdot \frac{r}{t} \tag{1}$$

As an aside note, the longitudinal stress in the cylinder wall is obtained by considering the force $p\pi r^2$ acting on the cylinder ends distributed over the area $2\pi rt$ of the cylinder walls. That gives a longitudinal stress equal to $pr/2t$, which is half the hoop stress observed in Eq. (1). This is why spherical pressure vessels are optimal, and it is why sausages break along their longitudinal axis when you boil them.

Non-uniform Cyclinders

Another cylinder case is visualized in Figure 2. Here, the ends are considered as a restraint on the ability of the cylinder to displace radially. A restraining belt in the middle is also introduced, having the same effect. Furthermore, the pressure is non-uniform. This represents an extension of the considerations that led to the basic hoop stress formula in Eq. (1). In fact, the present case is an introduction to shell theory. Shells carry load in their plane, and also in the out-of-plane direction. As with any structural problem, equations of equilibrium, material law, and kinematic compatibility are established in order to determine the governing differential equation.

Equilibrium Between External Load and Stress Resultants

To that end, consider the stress resultants displayed in Figure 3. Carefully observe the subscripts on the bending moments; this notation differs from some other document posted on this website. Because of axial symmetry, dM_ϕ=0. Also, there is no net axial force in the x-direction; N_x can be superimposed on the shell solution presented here. Moment equilibrium about the upper edge yields the well-known formula for shear in terms of moment: $V_x=dM_x/dx$. Of greater interest is equilibrium in the radial direction. The resultant pressure acting on the infinitesimal portion of the cylinder wall visualized in Figure 3 is pressure times area, i.e., $p \cdot dx \cdot (r \cdot d\phi)$. The cylinder wall is resisting that force by means of the net shear force $dV_x \cdot (r \cdot d\phi)$ and also the hoop stress, denoted by the stress resultant N_ϕ, which adds up to $dx \cdot N_\phi$ as a force resultant acting on that edge. Figure 4 shows in red colour the radial component of $dx \cdot N_\phi$, utilizing the fact that $d\phi$ in the geometry of the force polygon. In short, equilibrium in the radial direction reads

$$dV_x \cdot r \cdot d\phi + d\phi \cdot N_\phi \cdot dx = p \cdot dx \cdot r \cdot d\phi \tag{2}$$

Dividing through by $dx \cdot r \cdot d\phi$ yields

$$\frac{\mathrm{d}V_x}{\mathrm{d}x} + \frac{N_\phi}{r} = p(x) \tag{3}$$

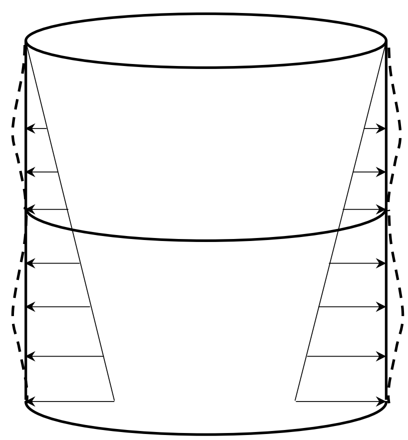

Figure 2: Nonuniform cylinder.

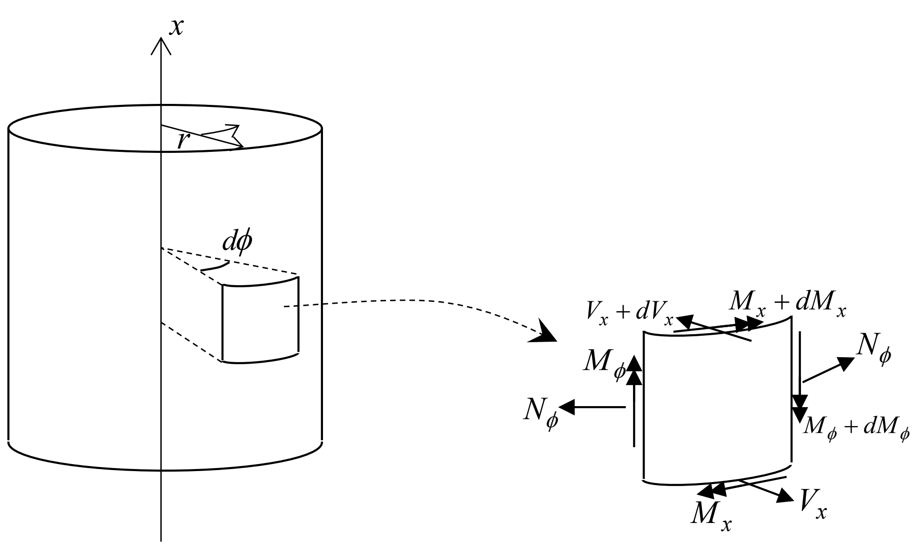

Figure 3: Stress resultants.

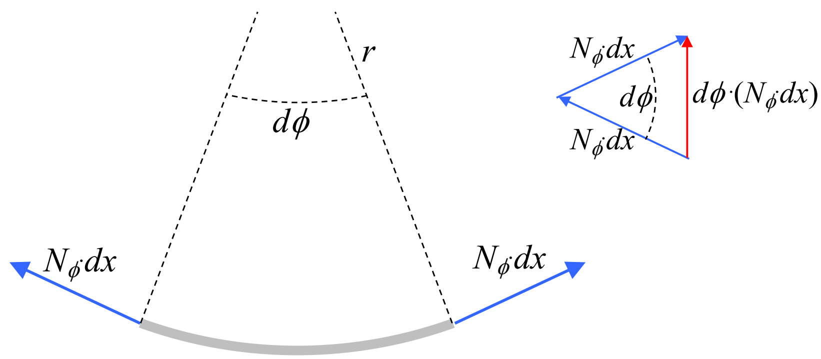

Figure 4: Radial component of hoop stress resultant.

Combined with the previously mention equation for V_x, Eq. (3) gives the result

$$\frac{\mathrm{d}^2 M_x}{\mathrm{d}x^2} + \frac{N_\phi}{r} = p(x) \tag{4}$$

Equilibrium Between Stress Resultants and Stress

The axial force in the hoop direction in the cylinder wall is

$$N_\phi = t \cdot \sigma_\phi \tag{5}$$

The bending moment about a vertical axis going through the cylinder wall is related to the axial as follows:

$$M_x = \int_{-t/2}^{t/2} \tau \cdot \sigma_x \, \mathrm{d}\tau \tag{6}$$

where τ is an auxiliary axis in the r-direction, having its origin at the midpoint of the cylinder wall.

Material Law

For the shell problem considered in this document, the material law must address strain in the x-direction, ε_x, accounting for bending about a horizontal longitudinal axis. Also, the material law must address strain in the ϕ-direction, i.e., ε_ϕ from hoop stress in the cylinder wall. However, notice that ε_ϕ does *not* account for any bending moment about a vertical axis; dM_ϕ=0 because of axial symmetry. The material law in the hoop direction is simply

$$\sigma_\phi = E\,\varepsilon_\phi \tag{7}$$

Our viewpoint now turns ninety degrees, for the consideration of bending about a horizontal axis following the circumference of the cylinder. That is associated with axial stress in the x-direction. Other vertical forces on the cylinder, i.e., acting in the x-direction, can be superimposed onto this shell solution. To address the bending of the shell, the considerations are similar to that of a plate solution, addressed elsewhere on this website. Because of the two-dimensional nature of the wall segment, Poisson's ratio enters:

$$\varepsilon_x = \frac{\sigma_x}{E} - \frac{\nu \cdot \sigma_\phi}{E} \tag{8}$$

Following the "plane strain" paradigm for this two-dimensional problem, the strain the circumferential direction is set equal to zero, which does not interfere with the previously established material law for hoop stress:

$$\varepsilon_\phi = \frac{\sigma_\phi}{E} - \frac{\nu \cdot \sigma_x}{E} = 0 \tag{9}$$

Combining Eq. (8) and (9) gives the following stress-strain relationship in the *x*-direction, recognized from plate theory:

$$\varepsilon_x = \frac{(1-\nu^2)}{E} \cdot \sigma_x \tag{10}$$

Kinematic Compatibility

The strain in the circumferential direction, i.e., the hoop direction is

$$\varepsilon_\phi = \frac{"\Delta L"}{"L"} = \frac{\big((r+w)\cdot d\phi\big) - \big(r \cdot d\phi\big)}{\big(r \cdot d\phi\big)} = \frac{w}{r} \tag{11}$$

where w is the displacement in the radial direction. The strain due to bending about the x-axis is, from basic beam theory:

$$\varepsilon_x = -\tau \cdot \frac{\mathrm{d}^2 w}{\mathrm{d}r^2} \tag{12}$$

Combining Equations

Combination of equations for stress resultant, material law, and kinematic compatibility in the hoop-direction, i.e., combination of Eqs. (5), (7), and (11) yields

$$N_\phi = t \cdot \sigma_\phi = t \cdot E \cdot \varepsilon_\phi = t \cdot E \cdot \frac{w}{r} \tag{13}$$

Combination of the same equations for the x-direction, i.e., Eqs. (6), (10), and (12) for the bending problem yields

$$M_x = -\int_{-t/2}^{t/2} \tau \cdot \sigma_x \, \mathrm{d}r = -\int_{-t/2}^{t/2} \tau \cdot \frac{E}{(1-\nu^2)} \cdot \varepsilon_x \, \mathrm{d}r = \int_{-t/2}^{t/2} \tau^2 \cdot \frac{E}{(1-\nu^2)} \cdot \frac{\mathrm{d}^2 w}{\mathrm{d}r^2} \, \mathrm{d}r = \frac{EI}{(1-\nu^2)} \cdot \frac{\mathrm{d}^2 w}{\mathrm{d}r^2} \tag{14}$$

Equilibrium with the externally applied load, i.e., the pressure within the cylinder, is now introduced by substituting Eqs. (13) and (14) into Eq. (4), which gives

$$\frac{EI}{(1-\nu^2)} \cdot \frac{\mathrm{d}^4 w}{\mathrm{d}r^4} + \frac{Et}{r^2} \cdot w = p(x) \tag{15}$$

Interestingly, this differential equation has exactly the same form as that of a beam on an elastic foundation. Physically, this is understood by thinking of a vertical strip of the cylinder wall, supported by the hoop stress effect. In other words, the hoop stress acts as an elastic support of that strip of the cylinder wall. By defining

$$k_{\text{shell}} = \frac{EI}{(1-\nu^2)} \tag{16}$$

the differential equation takes the form

$$\frac{\mathrm{d}^4 w}{\mathrm{d}r^4} + \frac{Et}{k_{\text{shell}} \cdot r^2} \cdot w = \frac{p(x)}{k_{\text{shell}}} \tag{17}$$

with

$$l_c = \sqrt[4]{\frac{4k_{\text{shell}} r^2}{Et}} \tag{18}$$

being the characteristic length known from the document on beams on elastic foundation, now for a cylinder. Further details about the solution are given in that document.