Cables and the Catenary Shape

Many situations exist in which a cable represents an efficient means of carrying loads. Some conceptual examples are shown in Figure 1. The application dictates the assumptions made about the cable:

- Neglect axial deformations in the cable?
- Neglect bending stiffness of the cable?
- Neglect the gravity load from the weight of the cable?
- Allow a large sag of the cable compare to the span width?

Several of those choices are explored in this document. Depending on the assumptions, the cable may or may not form the special catenary shape, which is associated with a cable with self-weight hanging freely. Some applications, assumptions, and consequences are here listed, with reference to Figure 1:

a) Neglect axial deformations, bending stiffness, and cable weight and consider only point loads, analyzed by equilibrium. Resulting shape: *Straight lines*.
b) Neglect axial deformations and bending stiffness, but include a uniformly distributed load, which may represent the cable weight, because here the sag is assumed to be very small. Resulting shape: *Parabola*.
c) Neglect axial deformations, bending stiffness, and cable weight, but include a uniformly distributed load and potentially large sag, including the possibility that the supports are not aligned along a horizontal line. Resulting shape: *Parabola*.
d) Neglect axial deformations and bending stiffness, but not the cable weight, and include potentially large sag, including the possibility that the supports are not aligned along a horizontal line. Resulting shape: *Catenary*.
e) Include bending stiffness and cable weight, and possibly axial deformations, in conjunction with potentially large sag; this is a more advanced theory with boundary effects, not addressed in this document.

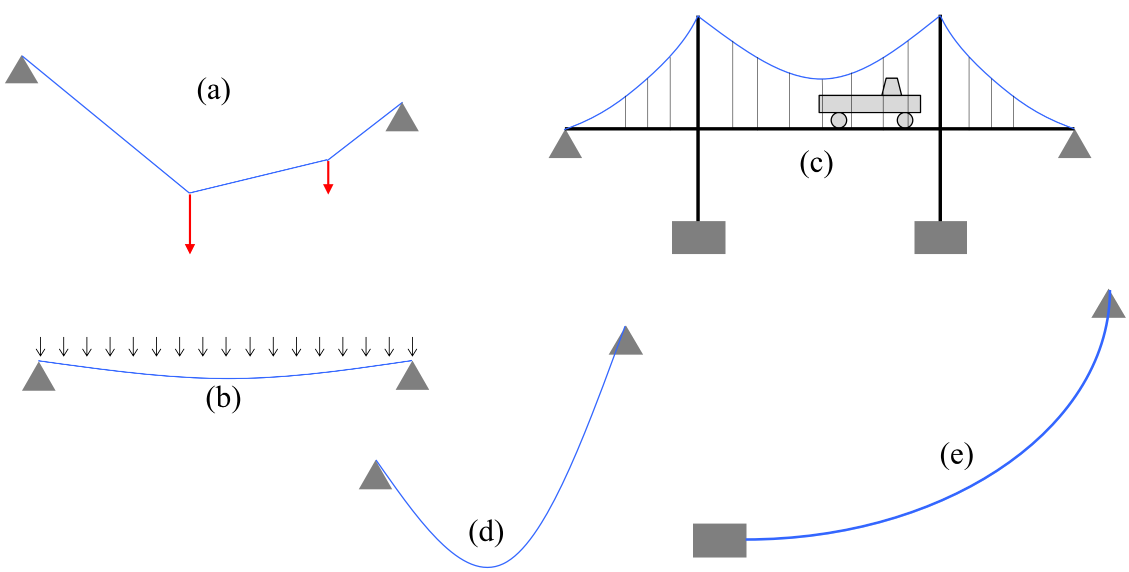

Figure 1: Cables, with c) and d) representing the catenary shape.

Case (a): Straight Lines

Equilibrium for cable that forms straight lines in the face of point loads is not considered in this document at this time.

Case (b): Parabola

In this case, visualized in Figure 1b), the sag is so small that we essentially consider the axial force in the cable, T, to be constant along the x-axis. That allows us to focus on equilibrium in the vertical direction, which is done in Figure 2. Notice that the slope, dy/dx, is negative on the left-hand side, and also that the slope changes from x to $x+dx$. Just like in the document on cylindrical shells, it is that *change in slope* that creates the vertical force resultant from T to resist the force $q \cdot dx$:

$$-T \cdot \frac{dy(x)}{dx} + T \cdot \frac{dy(x+dx)}{dx} = T \cdot \left(\frac{dy(x+dx)}{dx} - \frac{dy(x)}{dx} \right) = T \cdot \frac{d^2y(x)}{dx^2} \cdot dx = q \cdot dx \qquad (1)$$

That second-order differential equation is simply integrated twice, leading to a second-order equation for the cable, $y(x)$, with two integration constants. However, beyond suggesting a parabolic shape of the cable, the small sag and constant T limits the applicability of this solution.

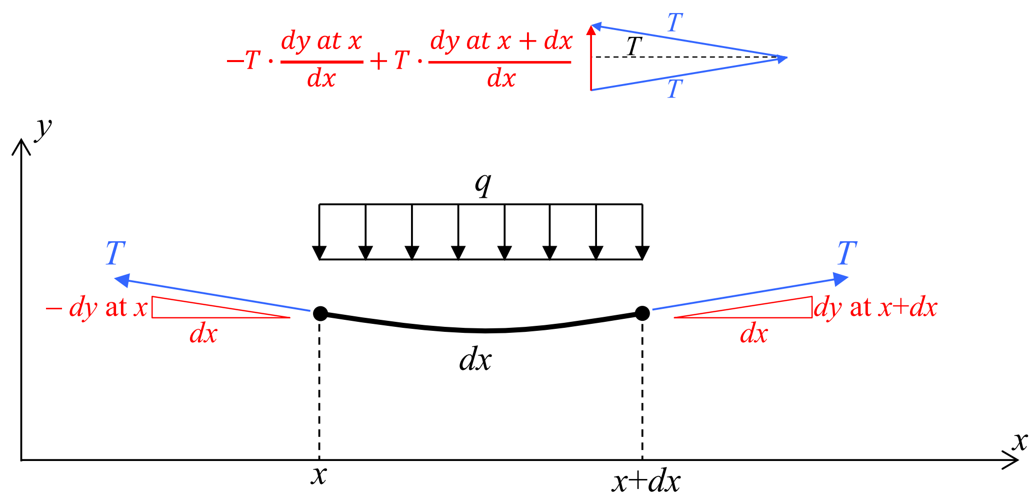

Figure 2: Vertical equilibrium.

Case (c): Parabola

Now consider a cable in which we neglect axial deformation, bending stiffness, and gravity load of the cable. However, we allow for the possibility of large sag and different vertical coordinates of the end points of the cable. However, the load must be vertical and uniformly distributed along the x-axis, as indicated in Figure 1c) and Figure 3. The objective in the following derivations is to derive an equation for $y(x)$, i.e., the vertical position of the cable for any given horizontal position, x. The derivations commence with equilibrium considerations. With reference to Figure 3, first consider the x-direction:

$$\sum F_x = -T \cdot \cos(\theta) + (T + dT) \cdot \cos(\theta + d\theta) = 0 \tag{2}$$

Next, consider the *y*-direction:

$$\sum F_y = -T \cdot \sin(\theta) - q \cdot dx + (T + dT) \cdot \sin(\theta + d\theta) = 0 \tag{3}$$

Then, moment equilibrium about A of Figure 3:

$$\sum M_A = -T \cdot \cos(\theta) \cdot dy + T \cdot \sin(\theta) \cdot dx = 0 \tag{4}$$

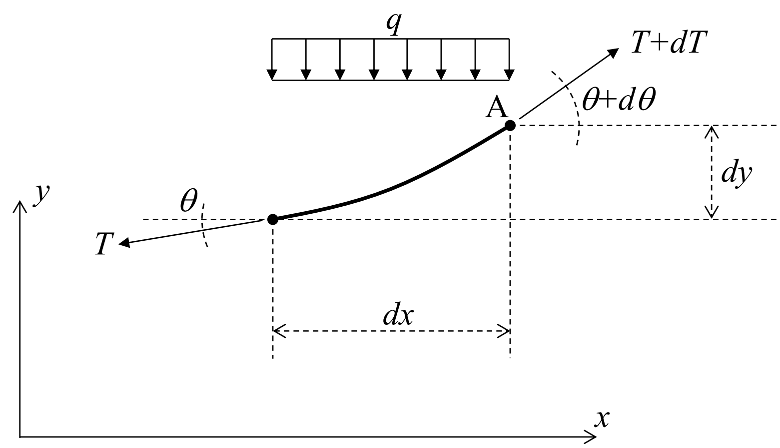

Figure 3: Equilibrium.

Each of those three equations are now rewritten. First, consider Eq. (2), which is reorganized to read

$$-T \cdot \cos(\theta) + T \cdot \cos(\theta + d\theta) + dT \cdot \cos(\theta + d\theta) = 0 \tag{5}$$

Then reorganize it to read

$$T \cdot (\cos(\theta + d\theta) - \cos(\theta)) + dT \cdot \cos(\theta + d\theta) = 0 \tag{6}$$

Then divide through by dx in order to obtain

$$T \cdot \frac{\cos(\theta + d\theta) - \cos(\theta)}{dx} + \frac{dT}{dx} \cdot \cos(\theta + d\theta) = 0 \tag{7}$$

Then recognize that $d\theta$ is small, which means that Eq. (7) becomes

$$T \cdot \frac{d(\cos(\theta))}{dx} + \frac{dT}{dx} \cdot \cos(\theta) = 0 \tag{8}$$

By the product rule of differentiation, that is the same as

$$\frac{d(T \cdot \cos(\theta))}{dx} = 0 \tag{9}$$

Similarly, equilibrium in the *y*-direction, expressed in Eq. (3), becomes

$$\frac{d(T \cdot \sin(\theta))}{dx} = q \tag{10}$$

Because $\sin(\theta)/\cos(\theta)=\tan(\theta)$, Eq. (4) represents the geometric relationship

$$\frac{dy}{dx} = \tan(\theta) \tag{11}$$

To summarize, Eqs. (2), (3), and (4) have been reorganized in order to become Eqs. (9), (10) and (11). Next, Eqs. (9) and (10) are integrated with respect to *x*, which yields

$$T \cdot \cos(\theta) = c_1 \tag{12}$$

and

$$T \cdot \sin(\theta) = q \cdot x + c_2 \tag{13}$$

where c_1 and c_2 are integration constants. Notice that, because Eq. (12) represents equilibrium in the *x*-direction, c_1 is the force in the *x*-direction in the cable. Interestingly, it is constant and does not depend on *x*. On the same note, observe that c_2 is the vertical force in the cable at $x=0$. Eqs. (12) and (13) are now combined by first solving for *T* in (12). The result is

$$c_1 \cdot \frac{\sin(\theta)}{\cos(\theta)} = q \cdot x + c_2 \tag{14}$$

Dividing through by c_1 and employing $\sin(\theta)/\cos(\theta)=\tan(\theta)$ in conjunction with Eq. (11) yields

$$\frac{dy}{dx} = \frac{q \cdot x}{c_1} + \frac{c_2}{c_1} \tag{15}$$

Finally, Eq. (15) is integrated in order to obtain the sought expression for $y(x)$:

$$y(x) = \frac{1}{2} \cdot \frac{q \cdot x^2}{c_1} + \frac{c_2}{c_1} \cdot x + c_3 \tag{16}$$

where c_3 is a new integration constant. Eq. (16) shows a parabolic shape of the cable, and the presence of three integration constants makes sense because three boundary conditions are required to specify a specific parabolic shape. The unknowns c_1, c_2, and c_3 can be obtained by specifying $y(0)$, $y(L)$, as well as the "sag," i.e., the value of *y* at the location where $dy/dx=0$. Because the cable is inextensible, that shape will remain, regardless of the magnitude of the load, q. In order to accommodate the third of the aforementioned boundary conditions, reconsider Eq. (15), which says that

$$x = -\frac{c_2}{q} \tag{17}$$

when dy/dx=0. As a result, these are boundary conditions consistent with the aforementioned three y-values, with H_1, H_2, and H_3 being the cable position at x equals 0, x equals zero slope, and x equals L, respectively:

- $y(0) = c_3 = H_1$
- $y\left(-\frac{c_2}{q}\right) = \frac{1}{2} \cdot \frac{q \cdot \left(-\frac{c_2}{q}\right)^2}{c_1} - \frac{c_2}{c_1} \cdot \frac{c_2}{q} + c_3 = H_2$
- $y(L) = \frac{1}{2} \cdot \frac{q \cdot L^2}{c_1} + \frac{c_2}{c_1} \cdot L + c_3 = H_3$

Solving those three equations in the three unknowns yields

$$c_1 = \frac{(H_1 - 2 \cdot H_2 + H_3) \cdot qL^2 + 2 \cdot \sqrt{\big((H_2 - H_1) \cdot (H_2 - H_3)\big) \cdot q^2 \cdot L^4}}{2 \cdot (H_1 - H_3)^2} \tag{18}$$

$$c_3 = \frac{(H_2 - H_1) \cdot qL^2 + \sqrt{\big((H_2 - H_1) \cdot (H_2 - H_3)\big) \cdot q^2 \cdot L^4}}{(H_1 - H_3) \cdot L} \tag{19}$$

$$c_3 = H_1 \tag{20}$$

Notice that Eqs. (18) and (19) cause problems when H_1=H_3, i.e., when the end points are at the same height. In that case, we know that the location of height H_2 is at the midpoint, and the solution is

$$c_1 = \frac{q \cdot L^2}{4(H_1 + H_3 - 2H_2)} \tag{21}$$

$$c_3 = \frac{4H_2qL - 3H_1qL - H_3ql}{4(H_1 + H_3 - 2H_2)} \tag{22}$$

$$c_3 = H_1 \tag{23}$$

Once c1, c2, and c3 for Eq. (16) have been determined directly from the input H_1, H_2, and H_3, the axial force in the cable is determined from Eqs. (11) and (12). That entails first finding the orientation, θ, at a given location:

$$\theta = \text{atan}\left(\frac{dy}{dx}\right) \tag{24}$$

Thereafter, the axial force in the cable at that location is

$$T = \frac{c_1}{\cos(\theta)} \tag{25}$$

This means that the distribution of axial force along the cable follows the slope of the cable, which is greatest at the end points and smallest where the cable has zero slope. In fact, the cable end that sits highest will be associated with the highest axial force, T. For the purposes of support design, recall from earlier that c_1 is the horizontal force anywhere in the cable. Furthermore, observe in Eq. (13) that the vertical force at any point in the cable is the absolute value of $q \cdot x + c_2$.

Case (d): Catenary

When considering a cable sagging solely under its own weight, neglecting axial deformations and bending stiffness, the catenary shape will emerge, instead of a parabola. However, several equations from earlier are useful. First, let the vertical distributed load be $\rho g \cdot s$, where ρ=mass density per unit length, g=acceleration of gravity, and s=axis measuring the length of a cable segment, with reference to Figure 4.

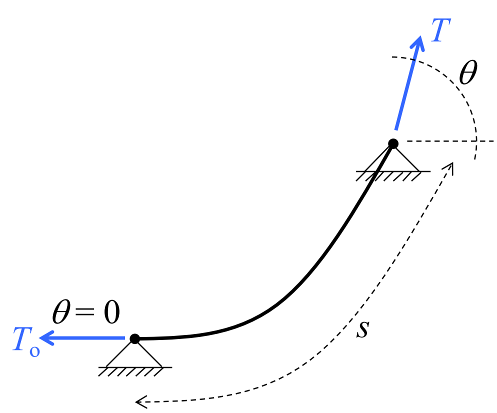

Figure 4: Definition of the *s*-coordinate.

Mimicking Eq. (12), horizontal equilibrium of the cable segment in Figure 4 yields

$$T \cdot \cos(\theta) = T_o \tag{26}$$

Similarly, mimicking Eq. (13), vertical equilibrium yields, because T_o is horizontal:

$$T \cdot \sin(\theta) = \rho g \cdot s \tag{27}$$

Solving Eq. (26) for T and substituting it into Eq. (27), recalling that $\sin(\theta)/\cos(\theta)=\tan(\theta)$, plus introducing the basic geometric relationship from Eq. (11) yields

$$\frac{dy}{dx} = \frac{\rho g}{T_o} \cdot s \tag{28}$$

where it is convenient to define the constant

$$c = \frac{T_o}{\rho g} \tag{29}$$

so that Eq. (28) reads

$$\frac{dy}{dx} = \frac{s}{c} \tag{30}$$

Thus far, these are replicas of developments in the previous section. The new developments relate to the coordinate *s*, which follows the cable. For an infinitesimally short portion of the cable, Pythagoras suggests

$$ds^2 = dx^2 + dy^2 \tag{31}$$

with reference to Figure 5.

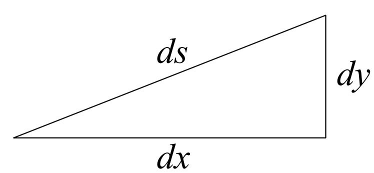

Figure 5: Relating *x*, *y*, and *s*.

Dividing through by dx^2 yields

$$\left(\frac{ds}{dx}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^2 \tag{32}$$

Substitution of Eq. (30) yields

$$\left(\frac{ds}{dx}\right)^2 = 1 + \left(\frac{s}{c}\right)^2 \tag{33}$$

By letting the number one in Eq. (33) be written c^2/c^2, that equation is rewritten

$$\left(\frac{ds}{dx}\right)^2 = \frac{c^2 + s^2}{c^2} \tag{34}$$

Taking the square root on both sides yields the governing differential equation for *s*:

$$\frac{ds}{dx} = \frac{\sqrt{c^2 + s^2}}{c} \tag{35}$$

Interestingly, that differential equation can be solved by rewriting it in terms of a new variable ξ. The parameter substitution that works is $s=c\cdot\sinh(\xi)$. That means the derivative in Eq. (35) is

$$\frac{ds}{dx} = \frac{ds}{d\xi} \cdot \frac{d\xi}{dx} = c \cdot \cosh(\xi) \cdot \frac{d\xi}{dx} \tag{36}$$

because the derivative of sinh(ξ) is cosh(ξ). That leads to the following result when substituted into Eq. (35):

$$c \cdot \cosh(\xi) \cdot \frac{d\xi}{dx} = \frac{\sqrt{c^2 + (c \cdot \sinh(\xi))^2}}{c} = \sqrt{1 + (\sinh(\xi))^2} \tag{37}$$

where the last equality recognizes that c cancels. Because of the mathematical identity

$$\frac{\cosh(\xi)}{\sqrt{1 + (\sinh(\xi))^2}} = 1 \tag{38}$$

Eq. (37) simplifies to

$$\frac{d\xi}{dx} = \frac{1}{c} \tag{39}$$

That is indeed a simpler differential equation than Eq. (35), solved by straightforward integration to give the solution

$$\xi = \frac{x}{c} + c_2 \tag{40}$$

where c_2 is another integration constant that vanishes if we let the origin of the x and s axes coincide. In short, the solution to the differential equation in Eq. (35) is $s=c\cdot\sinh(\xi)$ with ξ given in Eq. (40). In order to recover a solution in terms of $y(x)$ instead of $s(x)$, the relationship between y and s, provided earlier in Eq. (30), is brought to bear:

$$\frac{dy}{dx} = \frac{\mathrm{c} \cdot \sinh\left(\frac{x}{c}\right)}{c} = \sinh\left(\frac{x}{c}\right) \tag{41}$$

Integration gives the sought equation for the catenary shape:

$$y = \mathrm{c} \cdot \cosh\left(\frac{x}{c}\right) \tag{42}$$

That is a neat equation, but it actually hard, but possible, to calibrate c together with a lift and shift of the hyperbolic cosine shape for given values H_1, H_2, and H_3, as is done earlier in this document for the parabola. Instead, it is helpful to simply experiment with different parameter values and observing how the catenary appears. That said, it is of interest to calculate the axial force in the cable once a shape is set. To that end, consider the expression for the axial for T from Eq. (27), substitute $s=c\cdot\sinh(x/c)$ and θ=arctan(dy/dx) with dy/dx from Eq. (41) in order to obtain

$$T = \frac{\rho g \cdot s}{\sin(\theta)} = \frac{\rho g \cdot c \cdot \sinh\left(\frac{x}{c}\right)}{\sin(\theta)} = \frac{\rho g \cdot c \cdot \sinh\left(\frac{x}{c}\right)}{\sin\left(\arctan\left(\sinh\left(\frac{x}{c}\right)\right)\right)} = \rho g \cdot c \cdot \sqrt{1 + \sinh^2\left(\frac{x}{c}\right)} \qquad (43)$$

which simplifies to $T=\rho g \cdot y$ because of the mathematical identity in Eq. (38). The product $T \cdot \cos(\theta)$ gives the constant horizontal force component $\rho g \cdot c$. The product $T \cdot \sin(\theta)$ gives the vertical component $\rho g \cdot c \cdot \sinh(x/c)$. These results are demonstrated in an example posted near this document, including both a parabola and a catenary. Observe there that the constant c also represents the y-value of the catenary shape at x=0.