Stress Transformations, Mohr's Circle, Principal Stress

The stresses acting in the coordinate directions can be transformed into other directions by rotating the infinitesimally small material particle that the stresses act on. When Augustin Cauchy invented the concept of stress in 1822, he included the equilibrium considerations for a tetrahedron that are necessary to rotate the stresses. In 1882, Otto Mohr presented a graphical approach for the same, now known as Mohr's circle.

2D Transformations

Consider the plane stress state, in which only σ_{xx}, σ_{yy}, and $\sigma_{xy}=\sigma_{yx}\equiv\tau_{xy}$ act. Suppose these coordinate stresses are known. The objective in this section is to determine the stress state in rotated configurations, for example to determine the minimum and maximum axial stresses, i.e., the principal stresses. Let θ denote the angle (positive counter-clockwise) between the original coordinate system and the rotated one. The rotated plane is shown in Figure 1, where the stresses on the rotated plane are called σ and τ.

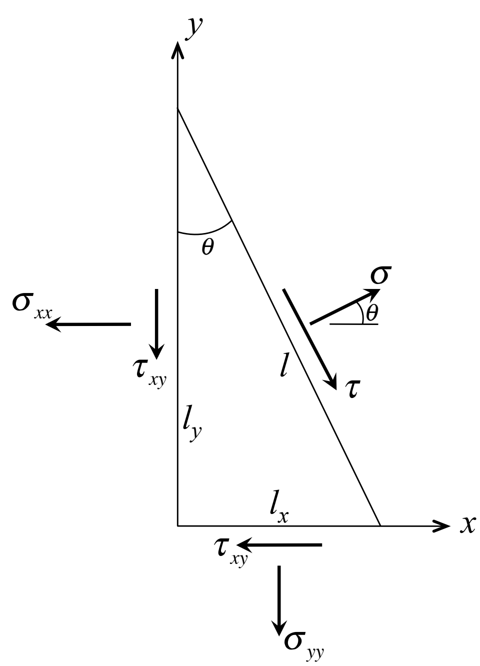

Figure 1: Stresses on an inclined plane.

By noting that $\cos(\theta)=l_y/l$ and $\sin(\theta)=l_x/l$, equilibrium in the direction of σ yields

$$\sigma = \sigma_{xx} \cdot \cos^2(\theta) + \sigma_{yy} \cdot \sin^2(\theta) + 2 \cdot \tau_{xy} \cdot \cos(\theta) \cdot \sin(\theta) \tag{1}$$

The trigonometric identities $\cos^2(\theta)=(1+\cos(2\theta))/2$, $\sin^2(\theta)=(1+\sin(2\theta))/2$, and $\sin(2\theta)=2\sin(\theta)\cos(\theta)$ lead to the modified expression

$$\sigma = \frac{\sigma_{xx} + \sigma_{yy}}{2} + \frac{\sigma_{xx} - \sigma_{yy}}{2} \cdot \cos(2\theta) + \tau_{xy} \cdot \sin(2\theta) \tag{2}$$

Similarly, equilibrium in the direction of τ yields:

$$\tau = \frac{\sigma_{xx} - \sigma_{yy}}{2} \cdot \sin(2\theta) - \tau_{xy} \cdot \cos(2\theta) \tag{3}$$

Eqs. (2) and (3) establish the basis for the transformation, i.e., rotation of stresses for two-dimensional stress states. Extreme values of σ and τ, plus the corresponding angle θ, are determined by setting the derivative of Eqs. (2) and (3) with respect to θ equal to zero. However, the graphical approach known as Mohr's circle is an appealing alternative to analytical derivations.

Mohr's Circle

Eqs. (2) and (3) represent a circle in the σ–τ plane. To derive the expression for the circle, move the first term in the right-hand side of Eq. (2) to the left-hand side. Then square Eqs. (2) and (3) and add them. Upon using the trigonometric identity $\sin^2(\theta)+\cos^2(\theta)=1$ and cancelling terms, one obtains

$$\left(\sigma - \frac{\sigma_{xx} + \sigma_{yy}}{2}\right)^2 + \tau^2 = \left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \tau_{xy}^2 \tag{4}$$

Comparing with $x^2+y^2=r^2$ suggests that Eq. (4) is a circle in the σ–τ plane, shifted along the σ–axis. The circle represented by Eq. (4) is called Mohr's circle, after Christian Otto Mohr's 1882 idea. All points on Mohr's circle represents stress states at planes of different angle θ.

Conventions

Mohr's circle is ingenious. However, its practical use is challenging because of the variety of sign conventions that can be adopted. The concept of a "pole" may or may not be used, and special attention is needed for the sign of the shear stress. This document adopts a specific set of choices, of many available, and those choices are reflected in the following procedure to draw & use Mohr's circle, with reference to Figure 2:

1. For the problem at hand, take note of the coordinate stresses, σ_{xx}, σ_{yy}, and τ_{xy}, at a specific point in the solid.
    a. Throughout, let axial stress be positive in tension and negative in compression.
    b. The coordinate shear stress, τ_{xy}, is positive if it is in the y-direction when acting on the surface whose normal vector is in the x-direction; this is the standard physical sign convention for shear stress. Note that, because $\tau_{yx}=\tau_{xy}$, the coordinate shear stress τ_{yx} is positive if it is in the x-direction when acting on the surface whose normal vector is in the y-direction
2. Calculate the radius and centre-shift of the circle implied by Eq. (4).
3. Draw Mohr's circle with the calculated radius and centre-shift, shown as a blue line in Figure 2.

4. On the circle, identify the points representing the stress state (σ_{xx}, τ_{xy}) and the stress state (σ_{yy}, τ_{xy}), now noting the following shear stress conventions:
    a. The point (σ_{xx}, τ_{xy}) should be plotted **below** zero if τ_{xy} is positive; to remember this, think of τ_{xy} being **negative** for clockwise shear in a beam laid along the *x*-axis. This point is red in Figure 2.
    b. The point (σ_{yy}, τ_{xy}) should be plotted **above** zero if τ_{xy} is positive; this point is blue in Figure 2.
5. From the point (σ_{xx}, τ_{xy}), which is on the circle, draw a **horizontal** line until it intersects with the circle again; that point is the "pole;" alternatively, draw a **vertical** line from (σ_{yy}, τ_{xy}) if you want to study σ_{yy} instead of σ_{xx}.
6. From the pole point, draw lines in any direction; the point at which the line intersects the circle is a stress state with the following meaning:
    a. The orientation of the line is the orientation of the square on which stresses act.
    b. Only pay attention to the stress on the "near & far" edges of the square at the end of each line, as shown in Figure 2.
    c. If the intersection point is on the positive part of the σ-axis then the axial stress on the near & far edges is tension.
    d. If the point on the circle is on the positive part of the τ-axis then the shear stress on the near & far edges twists the square clockwise.

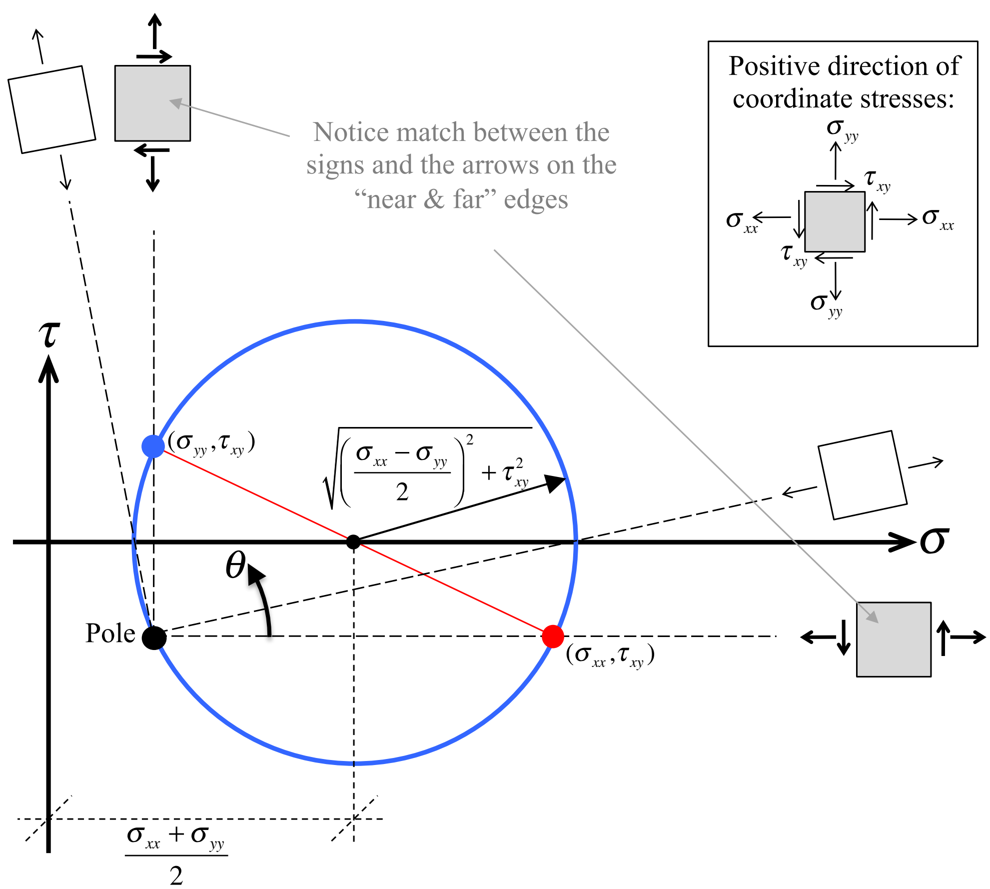

Figure 2: Mohr's circle in blue.

Looking at Mohr's circle, the following observations are made:

- The stress state (σ_{xx}, τ_{xy}) is at a horizontal line from the pole point; this is the stress state that acts on the plane that has the *x*-axis as the surface normal
- The stress state (σ_{yy}, τ_{xy}) is at a vertical line from the pole point; this is the stress state that acts on the plane that has the *y*-axis as the surface normal
- Drawing the circle immediately reveals the maximum and minimum axial stress; they appear at locations, i.e., orientations with zero shear stress
- The stress states with maximum shear stress are usually *not* associated with zero axial stress
- When σ_{xx}=σ_{yy} the blue point coincides with the red point

Finally, for the 2D case, notice that's Mohr's circle implies that the maximum shear stress is

$$\tau_{\max} = \frac{1}{2}(\sigma_1 - \sigma_2) = \sqrt{\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \tau_{xy}^2} \tag{5}$$

3D Transformations

Consider the stress traction $\mathbf{t}=\{t_x\ t_y\ t_z\}^T$ that acts on an infinitesimal surface area with surface normal $\mathbf{n}=\{n_x\ n_y\ n_z\}^T$ shown in Figure 3, where t_i is the force in the *i*-direction. Let dA denote the area of the inclined surface on which the traction acts and let dA_x denote the area of the side that has the negative *x*-axis as normal vector, and so forth. Equilibrium in the *x*-direction yields:

$$t_x \cdot dA = \sigma_{xx} \cdot dA_x + \sigma_{yx} \cdot dA_y + \sigma_{zx} \cdot dA_z \tag{6}$$

To refine the expression, consider the relationship between the areas dA and dA_i. Figure 3 shows that dA=0.5hl and dA_z=0.5$h_z l$.

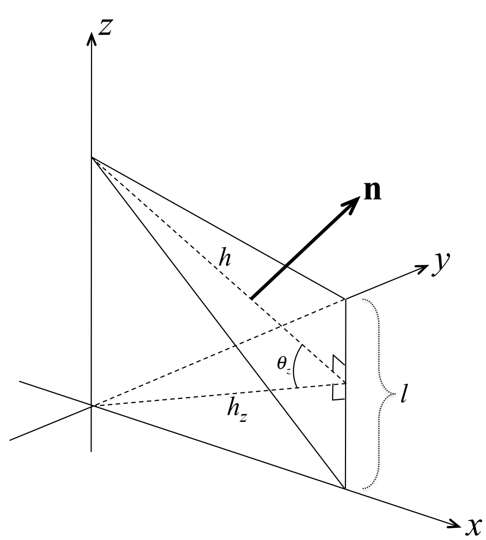

Figure 3: Surface on which the stress traction acts.

Consequently,

$$\frac{dA_z}{dA} = \frac{h_z}{h} = \cos(\theta_z) = \cos(z, \mathbf{n}) = n_z \tag{7}$$

Dividing Eq. (6) by dA yields

$$t_x = \sigma_{xx} \cdot n_x + \sigma_{yx} \cdot n_y + \sigma_{zx} \cdot n_z \tag{8}$$

Repeating this exercise for all three axis-directions, and noting that $\sigma_{ij}=\sigma_{ji}$ because of equilibrium considered later in this document, yields the equilibrium equations derived by Cauchy that relate a surface traction to the coordinate stresses:

$$\mathbf{t} = \boldsymbol{\sigma}\mathbf{n} = \begin{Bmatrix} t_x \\ t_y \\ t_z \end{Bmatrix} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} \begin{Bmatrix} n_x \\ n_y \\ n_z \end{Bmatrix} = t_i = \sigma_{ij} n_j \tag{9}$$

It is noted that, because **n** is a unit vector, the axial stress on a plane with normal vector **n** is the dot product between **n** and the stress traction:

$$\sigma_{\mathbf{n}} = \mathbf{t}^T \mathbf{n} \tag{10}$$

Subsequently, the Pythagorean theorem determines the largest shear stress on the plane:

$$\tau_{\mathbf{n}} = \sqrt{\|\mathbf{t}\|^2 + \sigma_{\mathbf{n}}^2} \tag{11}$$

Principal Stresses

The axial stress acting on a plane with zero shear stress is called a principal stress. The principal stresses will always include the minimum and maximum possible axial stresses. One way of determining principal stresses for a 2D stress-state is to draw Mohr's circle. Referring to another document on Mohr's circle, the points on the circle crossing the abscissa axis, i.e., the axial stress values at locations with zero shear stress, are principal stresses. The direction of the principal axes is identified by drawing a straight line from the pole point to the locations on the circle with zero shear stress. Mathematically, the maximum axial stress is

$$\sigma_1 = \max\left\{0 \;\;,\;\; \frac{\sigma_{xx}+\sigma_{yy}}{2} + \sqrt{\left(\frac{\sigma_{xx}-\sigma_{yy}}{2}\right)^2 + \tau_{xy}^2}\right\} \tag{12}$$

and that the minimum axial stress is

$$\sigma_3 = \min\left\{0 \;\;,\;\; \frac{\sigma_{xx}+\sigma_{yy}}{2} - \sqrt{\left(\frac{\sigma_{xx}-\sigma_{yy}}{2}\right)^2 + \tau_{xy}^2}\right\} \tag{13}$$

Here, σ_1 and σ_3 are symbols reserved for the maximum and minimum stress, respectively. This explains the inclusion of zero as a possibility in Eqs. (12) and (13). The notation requires particular attention in 2D stress situations, where the out-of-plane stress is zero and, thus, often is σ_3. Another approach for determining principal stresses is to employ the following equilibrium, equation, derived by considering a tetrahedron in another document:

$$\mathbf{t} = \boldsymbol{\sigma}\mathbf{n} = \begin{Bmatrix} t_x \\ t_y \\ t_z \end{Bmatrix} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} \begin{Bmatrix} n_x \\ n_y \\ n_z \end{Bmatrix} = t_i = \sigma_{ij} n_j \tag{14}$$

For a plane with principal stresses the traction vector is parallel with the normal vector of that plane. That means there are no shear stresses on that plane and the traction is the scaled normal vector:

$$\mathbf{t} = \boldsymbol{\sigma}\mathbf{n} = \lambda \cdot \mathbf{n} \tag{15}$$

This is an eigenvalue problem in the unknown scalar λ, *i.e.*, $(\boldsymbol{\sigma} - \lambda \cdot \mathbf{I})\mathbf{n} = 0$. Solutions are obtained by setting the determinant of the coefficient matrix equal to zero:

$$\lambda^3 - I_1 \cdot \lambda^2 + I_2 \cdot \lambda - I_3 = 0 \tag{16}$$

where the stress invariants are defined as

$$I_1 = \sigma_{xx} + \sigma_{yy} + \sigma_{zz} \tag{17}$$

$$I_2 = \begin{vmatrix} \sigma_{yy} & \sigma_{yz} \\ \sigma_{zy} & \sigma_{zz} \end{vmatrix} + \begin{vmatrix} \sigma_{xx} & \sigma_{xz} \\ \sigma_{zx} & \sigma_{zz} \end{vmatrix} + \begin{vmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{yx} & \sigma_{yy} \end{vmatrix} \tag{18}$$

$$I_3 = |\boldsymbol{\sigma}| \tag{19}$$

where vertical bars indicate the determinant operation. Upon solving for the eigenvalues, λ, i.e., σ_1, σ_2, and σ_3, the eigenvectors yield the principal directions. The quantities I_1, I_2, and I_3 are called stress invariants because they retain the same value regardless of the orientation of the coordinate system. These stress invariants are somewhat different from the stress invariants J_1, J_2, and J_3 for the "deviatoric" stress tensor mentioned in the document on stress-based failure criteria and used in "J_2 plasticity."