Material Law

The objective of the material law, sometimes referred to as constitutive equations, is to define the relationship between stresses and strains in the material.

1D Hooke's Law

The uniaxial case is devoid of shear stress and shear strain. Hooke's law is then simply

$$\sigma = E \cdot \varepsilon \tag{1}$$

where σ=axial stress, E=modulus of elasticity, i.e., Young's modulus, and ε=axial strain.

2D Hooke's Law

A biaxial stress state requires the consideration of transversal strain, as well as shear strains. Transversal strain means strain in the y-direction when the material is pulled in the x-direction. Specifically, the axial strain in the y-direction has two contributions:

$$\varepsilon_{yy} = \frac{\sigma_{yy}}{E} - \nu \cdot \frac{\sigma_{xx}}{E} \tag{2}$$

where the first term emanates directly from Eq. (1) and the second term is strain in the y-direction caused by stress in the x-direction. The material constant that governs the latter, i.e., the transversal strain, is called Poisson's ratio, denoted by ν. That material-specific constant joins E in affecting the relationship between shear stress and shear strain as well. To understand that, consider the infinitesimally small two-dimensional material particle subjected to pure shear τ, shown in Figure 1. Mohr's circle for this case is centred at the origin with radius τ. Consequently, the principal stresses are $-\tau$ and τ with axes at 45°.

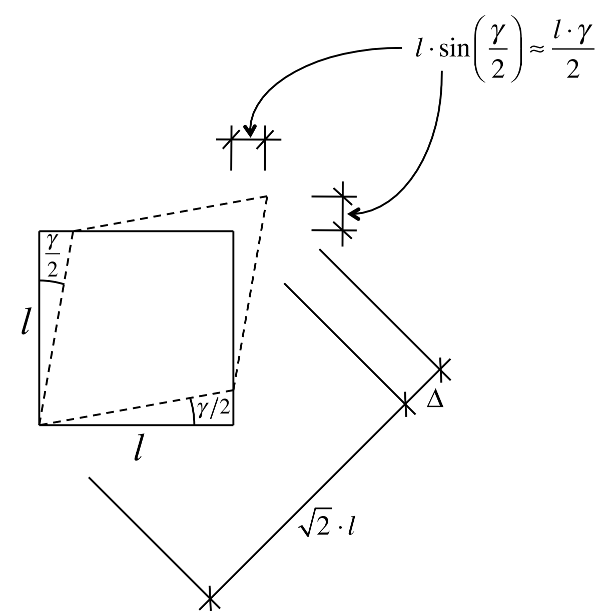

Figure 1: Derivation of the expression *G*.

The quantity Δ can be expressed in two ways. In the pure shear state:

$$\Delta = \sqrt{2 \cdot \left(\frac{l \cdot \gamma}{2} \right)^2} = \frac{l \cdot \gamma}{\sqrt{2}} \tag{3}$$

In the rotated state of pure axial stress:

$$\Delta = \varepsilon \cdot \left(\sqrt{2} \cdot l \right) = \left(\frac{\tau}{E} - \nu \cdot \frac{(-\tau)}{E} \right) \cdot \left(\sqrt{2} \cdot l \right) \tag{4}$$

Equating the two expressions for Δ yields:

$$\tau = \underbrace{\left(\frac{E}{2 \cdot (1+\nu)} \right)}_{\equiv G} \cdot \gamma \tag{5}$$

where the shear modulus, G, is defined, containing both E and ν.

Plane Stress

Hooke's law for 2D continuum problems has two versions: plane stress and plane strain. Plane stress applies when there are no stresses perpendicular to the considered plane. This applies to thin planar members that do not have stresses imposed perpendicular to the surface. One example is a deep beam that stretches along the x-axis with z as the vertical axis. Seen from the side this beam forms a 2D continuum problem in the x-z-plane with plane stress material law because there is no stress perpendicular to the sides of the beam. Conversely, plane strain implies that there is zero strain perpendicular to the plane under consideration. This applies to problems where the plane is a cross-section of a structure that is long in the direction perpendicular to that plane. One example is the modelling of the cross-section of a dam structure. From the selected cross-section the dam stretches out to both sides until it meets mountainside supports. This restrains displacement, and hence strain, perpendicular to the x-y-plane That plane is then in a state of plane strain, i.e., without strain perpendicular to the plane. The plane stress version of the material law is the straightforward simplification of the material law for 3D problems when one coordinate is removed:

$$\begin{aligned} \varepsilon_{xx} &= \frac{\sigma_{xx}}{E} - \nu \cdot \frac{\sigma_{yy}}{E} \\ \varepsilon_{yy} &= \frac{\sigma_{yy}}{E} - \nu \cdot \frac{\sigma_{xx}}{E} \\ \tau_{xy} &= G \cdot \gamma_{xy} \end{aligned} \tag{6}$$

where $G = E / \left(2 \cdot (1+\nu) \right)$. In Voight notation, the 2D material law is written generically in index and vector notation as follows:

$$\sigma_i = D_{ij} \cdot \varepsilon_j \quad \Leftrightarrow \quad \boldsymbol{\sigma} = \mathbf{D} \boldsymbol{\varepsilon} \tag{7}$$

where the **D**-matrix for plane stress is

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} = \underbrace{\frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \dfrac{1-\nu}{2} \end{bmatrix}}_{\mathbf{D}} \cdot \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} \tag{8}$$

or inversely, directly from Eq. (6):

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & 0 \\ -\nu & 1 & 0 \\ 0 & 0 & 2(1+\nu) \end{bmatrix} \cdot \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{Bmatrix} \tag{9}$$

Plane Strain

When there is zero strain perpendicular to the *x*-*y*-plane then the material law reads

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \dfrac{1-2\nu}{2} \end{bmatrix} \cdot \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} \tag{10}$$

or inversely

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} = \frac{1}{E} \begin{bmatrix} 1-\nu^2 & -\nu(1+\nu) & 0 \\ -\nu(1+\nu) & 1-\nu^2 & 0 \\ 0 & 0 & 2(1+\nu) \end{bmatrix} \cdot \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} \tag{11}$$

3D Hooke's Law

Considering all three axis directions, Hooke's law for axial strains reads

$$\begin{aligned} \varepsilon_{xx} &= \frac{\sigma_{xx}}{E} - \nu \cdot \frac{\sigma_{yy}}{E} - \nu \cdot \frac{\sigma_{zz}}{E} \\ \varepsilon_{yy} &= \frac{\sigma_{yy}}{E} - \nu \cdot \frac{\sigma_{xx}}{E} - \nu \cdot \frac{\sigma_{zz}}{E} \\ \varepsilon_{zz} &= \frac{\sigma_{zz}}{E} - \nu \cdot \frac{\sigma_{xx}}{E} - \nu \cdot \frac{\sigma_{yy}}{E} \end{aligned} \tag{12}$$

and for shear strains it reads

$$\tau_{xy} = G \cdot \gamma_{xy} \quad , \qquad \tau_{yz} = G \cdot \gamma_{yz} \quad , \qquad \tau_{zx} = G \cdot \gamma_{zx} \tag{13}$$

In Voight notation it reads $\varepsilon_i = C_{ij}^{-1} \sigma_j$ or $\boldsymbol{\varepsilon} = \mathbf{C}^{-1} \boldsymbol{\sigma}$:

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix} = \frac{1}{E} \cdot \left[\begin{array}{ccc|ccc} 1 & -\nu & -\nu & 0 & 0 & 0 \\ -\nu & 1 & -\nu & 0 & 0 & 0 \\ -\nu & -\nu & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 2(1+\nu) & 0 & 0 \\ 0 & 0 & 0 & 0 & 2(1+\nu) & 0 \\ 0 & 0 & 0 & 0 & 0 & 2(1+\nu) \end{array} \right] \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{Bmatrix} \tag{14}$$

Or, inversely $\sigma_i = C_{ij}\varepsilon_j$ or $\boldsymbol{\sigma} = \mathbf{C}\boldsymbol{\varepsilon}$:

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \cdot \left[\begin{array}{ccc|ccc} (1-\nu) & \nu & \nu & 0 & 0 & 0 \\ \nu & (1-\nu) & \nu & 0 & 0 & 0 \\ \nu & \nu & (1-\nu) & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} \end{array} \right] \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix} \tag{15}$$

In index notation with the original strain and stress tensors, Hooke's law is written

$$\sigma_{ij} = \lambda \cdot \varepsilon_{kk} \cdot \delta_{ij} + 2 \cdot \mu \cdot \varepsilon_{ij} \tag{16}$$

where δ_{ij} is the unit matrix and λ and μ are the Lame parameters:

$$\mu = G \qquad \lambda = \frac{\nu E}{(1+\nu)(1-2\nu)} \tag{17}$$

In addition to E, ν, G, μ, and λ, the bulk modulus, K, is employed in the study of volume change under hydrostatic pressure. Let $\varepsilon_{kk}=\varepsilon_{11}+\varepsilon_{22}+\varepsilon_{33}$ denote the dilatation, i.e., the change in volume of an infinitesimally small cube. The pressure, p, is $\varepsilon_{kk}/3$. The bulk modulus relates the pressure to the dilatation: $p=-K \cdot \varepsilon_{kk}$, where

$$K = \frac{E}{3(1-2\nu)} \tag{18}$$

References

Timoshenko, S., and Goodier, J. N. (1969). *Theory of elasticity*. McGraw-Hill.