Mass Matrix for Truss and Frame Elements

The standard procedure for establishing the stiffness matrix with shape functions can be applied to establish the mass matrix as well. That procedure first establishes the "weak form" of the boundary value problem from the "strong form," or from the principle of virtual displacements. Thereafter, shape functions are substituted and integration is carried out. The document on Vibration of Distributed Mass posted elsewhere on this website gives the strong form, i.e., differential equation, as

$$q = EI \cdot w'''' + \rho \cdot A \cdot \ddot{w} \tag{1}$$

where ρ=mass density and A=cross-section area. Each prime means one derivative with respect to x and each dot means one derivative with respect to time. Weighting and integrating yields the "weighted residual form" of the boundary value problem:

$$\int_0^L \left(\rho \cdot A \cdot \ddot{w} + EI \cdot w'''' - q \right) \cdot \delta w \cdot dx = 0 \tag{2}$$

Following the standard procedure, integration by parts yields

$$\int_0^L \left(\rho \cdot A \cdot \ddot{w} \cdot \delta w + EI \cdot w'' \cdot \delta w'' - q \cdot \delta w \right) \cdot dx = 0 \tag{3}$$

Substitution of the discretizations w=$\mathbf{Nu}$ and δw=$\mathbf{N}\delta\mathbf{u}$ and demanding arbitrary variations $\delta\mathbf{u}$ gives

$$\underbrace{\left[\int_0^L \rho \cdot A \cdot \mathbf{N}^T \mathbf{N} \, dx \right]}_{\mathbf{M}} \ddot{\mathbf{u}} + \underbrace{\left[\int_0^L EI \cdot \mathbf{N}''^T \mathbf{N}'' \, dx \right]}_{\mathbf{K}} \mathbf{u} = \underbrace{\left\{ \int_0^L q \cdot \mathbf{N}^T \, dx \right\}}_{\mathbf{F}} \tag{4}$$

where **M**=mass matrix, **K**=stiffness matrix, and **F**=load vector. **K** and **F** are addressed in the document on Euler-Bernoulli beam elements. In that document, the stiffness matrix is first developed in the Basic element configuration. That can be done here as well, for the mass matrix, but it is arbitrarily selected to start in the Local configuration, where the shape functions are (corresponding to the degrees of freedom are shown in Figure 1):

$$\begin{aligned} N_1(x) &= \frac{2x^3}{L^3} - \frac{3x^2}{L^2} + 1 \\ N_2(x) &= -\frac{x^3}{L^2} + \frac{2x^2}{L} - x \\ N_3(x) &= -\frac{2x^3}{L^3} + \frac{3x^2}{L^2} \\ N_4(x) &= -\frac{x^3}{L^2} + \frac{x^2}{L} \end{aligned} \tag{5}$$

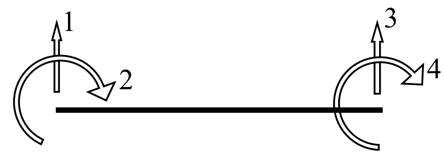

Figure 1: Local degrees of freedom.

Employing the shape functions in Eq. (5) to evaluate the expression for **M** in Eq. (4) yields the following mass matrix in the local element configuration:

$$\mathbf{M} = \rho A \cdot \begin{bmatrix} \frac{13\,L}{35} & -\frac{11\,L^2}{210} & \frac{9\,L}{70} & \frac{13\,L^2}{420} \\ -\frac{11\,L^2}{210} & \frac{L^3}{105} & -\frac{13\,L^2}{420} & -\frac{L^3}{140} \\ \frac{9\,L}{70} & -\frac{13\,L^2}{420} & \frac{13\,L}{35} & \frac{11\,L^2}{210} \\ \frac{13\,L^2}{420} & -\frac{L^3}{140} & \frac{11\,L^2}{210} & \frac{L^3}{105} \end{bmatrix} \tag{6}$$

In the Basic element configuration, only the second and the fourth degrees of freedom appear. As a result, the use of only the shape functions N_2 and N_4 give the following mass matrix in the Basic element configuration:

$$\mathbf{M} = \rho A \cdot \begin{bmatrix} \frac{L^3}{105} & -\frac{L^3}{140} \\ -\frac{L^3}{140} & \frac{L^3}{105} \end{bmatrix} \tag{7}$$

For an element that remains straight, i.e., a truss element, the shape functions associated with the first and third degrees of freedom in Figure 1 are

$$\begin{aligned} N_1(x) &= 1 - \frac{x}{L} \\ N_2(x) &= \frac{x}{L} \end{aligned} \tag{8}$$

Employing these shape functions to evaluate the expression for **M** in Eq. (4) yields the following mass matrix for a truss element:

$$\mathbf{M} = \rho A \cdot \begin{bmatrix} \frac{L}{3} & \frac{L}{6} \\ \frac{L}{6} & \frac{L}{3} \end{bmatrix} \tag{9}$$