Kinematic Compatibility

The objective with kinematic compatibility equations is to express internal strains in terms of external displacements and rotations. In this document, the matter is made out to be simpler than it is in more general continuum mechanics formulations. Herein, strain expressions will appear unique, albeit somewhat complex for 2D and 3D problems. However, in general, and especially for finite strains, choices must be made as to how the strain is expressed. That is addressed in the document on continuum mechanics formulated with tensors.

1D Kinematic Compatibility

While recognizing that this is a choice amongst several, the classical expression for infinitesimal strain is the ratio "change in length" to "original length:"

$$\varepsilon = \frac{du}{dx} \tag{1}$$

The ingredients in Eq. (1) are visualized in Figure 1; u denotes the displacement in the x-direction, which is the longitudinal direction of an infinitesimally small element of length dx.

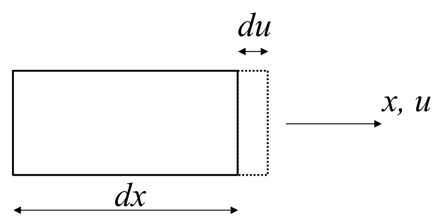

Figure 1: Ingredients in uniaxial strain.

2D Kinematic Compatibility

Shear strain is uncoupled from axial strain. A displacement field can simultaneously cause both shear strain and axial strain, but one does not cause the other. For that reason, the shear strain γ_{xy} is studied in isolation in Figure 2. γ_{xy} is the change in angle between originally orthogonal lines, counting two contributions:

$$\gamma_{xy} = \gamma_{xy} = \varepsilon_{xy} + \varepsilon_{yx} = \frac{\left(\frac{\partial v}{\partial x}\right)\cdot dx}{dx} + \frac{\left(\frac{\partial u}{\partial y}\right)\cdot dy}{dy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \tag{2}$$

With that expression, as well as the two uncoupled axial strains, the 2D kinematic compatibility equations in Voight notation read

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} \partial/\partial x & 0 \\ 0 & \partial/\partial y \\ \partial/\partial y & \partial/\partial x \end{bmatrix} \begin{Bmatrix} u \\ v \end{Bmatrix} \tag{3}$$

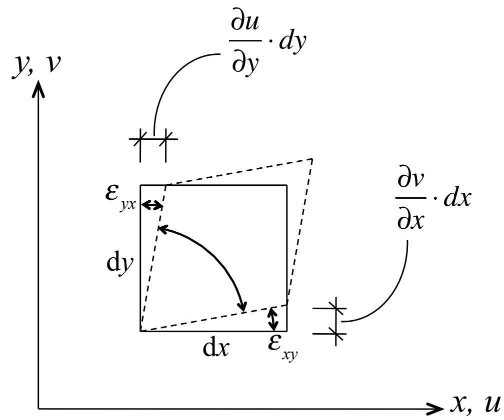

Figure 2: Shear strain.

Eq. (3) is often employed as the final compatibility equations in 2D problems. However, a further step can be taken. In Eq. (3), differentiating the first equation twice with respect to *y*, then differentiating the second equation twice with respect to *x*, and finally differentiating the third equation with respect to *x* and *y* yields

$$\frac{\partial^2 \varepsilon_{xx}}{\partial y^2} = \frac{\partial^3 u}{\partial x \partial y^2} \tag{4}$$

$$\frac{\partial^2 \varepsilon_{yy}}{\partial x^2} = \frac{\partial^3 v}{\partial y \partial x^2} \tag{5}$$

$$\frac{\partial^2 \gamma_{xx}}{\partial y \partial x} = \frac{\partial^3 u}{\partial y^2 \partial x} + \frac{\partial^3 v}{\partial x^2 \partial y} \tag{6}$$

Substituting Eqs. (4) and (5) into Eq. (6) yields the "compatibility equation" in 2D elasticity theory, which states the necessary relationship between the strains for a deformation pattern to be physically valid:

$$\frac{\partial^2 \gamma_{xy}}{\partial x \partial y} = \frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} \tag{7}$$

3D Kinematic Compatibility

When including all three axis directions, the axial strains are

$$\varepsilon_{xx} = \frac{\partial u}{\partial x}$$
$$\varepsilon_{yy} = \frac{\partial v}{\partial y} \quad (8)$$
$$\varepsilon_{zz} = \frac{\partial w}{\partial z}$$

and the shear strains are

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$
$$\gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \quad (9)$$
$$\gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}$$

By means of Voight notation those kinematic compatibility equations can be written in terms of vectors and matrices, with ∇ here defined as a differential operator:

$$\boldsymbol{\varepsilon} = \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix} = \begin{bmatrix} \partial/\partial x & 0 & 0 \\ 0 & \partial/\partial y & 0 \\ 0 & 0 & \partial/\partial z \\ \partial/\partial y & \partial/\partial x & 0 \\ 0 & \partial/\partial z & \partial/\partial y \\ \partial/\partial z & 0 & \partial/\partial x \end{bmatrix} \begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = \begin{bmatrix} u_{,x} \\ v_{,y} \\ w_{,z} \\ u_{,y} + v_{,x} \\ v_{,z} + w_{,y} \\ u_{,z} + w_{,x} \end{bmatrix} = \nabla \mathbf{u} \quad (10)$$

In index notation, the same equations are written as

$$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) \quad (11)$$

with summation implied over repeated indices.

References

Timoshenko, S., and Goodier, J. N. (1969). *Theory of elasticity*. McGraw-Hill.