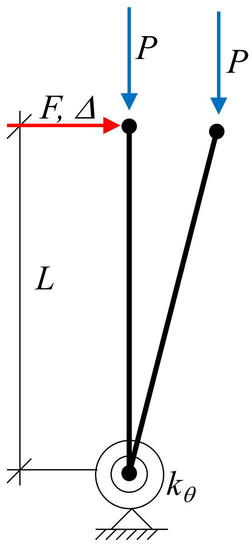
Inelastic Stick Model with P-delta

Consider a vertical cantilevered "stick model" of a column. It has a rotational spring at the bottom, and a lateral force F at the top, plus an axial force P at the top. The lateral displacement at the top is denoted Δ. The length of the column is L and its rotation is denoted θ.



Linear Elastic

Equilibrium:

$$M = F\,L$$

Material law:

$$M = k_\theta\,\theta$$

Kinematic compatibility:

$$\Delta = \theta\,L$$

Combine the equilibrium, material law, and kinematic compatibility to obtain the governing F-Δ relationship:

$$F = \frac{M}{L} = \frac{k_\theta\,\theta}{L} = \frac{k_\theta}{L^2}\,\Delta$$

where the linear elastic stiffness is identified.

Linear Material with P-delta

Equilibrium of the displaced shape includes the P-delta effect:

$$M = F L + P \Delta$$

The other governing equations remain the same as for the linear elastic case. Again combining equilibrium, material law, and kinematic compatibility to obtain the governing F-Δ relationship yields:

$$F = \frac{M}{L} - \frac{P}{L}\Delta = \frac{k_\theta \theta}{L} - \frac{P}{L}\Delta = \left(\frac{k_\theta}{L^2} - \frac{P}{L}\right)\Delta$$

Setting the stiffness equal to zero yields the buckling value of the axial force, P:

$$P_{cr} = \frac{k_\theta}{L}$$

Aside note: Calibrating the spring stiffness

One way in which to select a value for k_θ is to demand that the displacement at the tip of the stick model, subjected to the load F, should match the displacement of an elastic cantilevered beam, which is $\Delta = \frac{F \cdot L^3}{3 \cdot E \cdot I}$. As a result, the examination of the linear elastic stiffness from above, which is $\frac{k_\theta}{L^2}$, suggests that $k_\theta = \frac{3 \cdot E \cdot I}{L}$ would give the same displacement of the stick model and an elastic cantilever, subjected to a lateral load F. Subsitution of that spring stiffness into the previously determined buckling load gives:

$$P_{\text{cr}} = \frac{3 \cdot E \cdot I}{L^2}$$

In contrast, the exact Euler buckling load for a cantilever is $P_{\text{cr}} = \frac{\pi^2 \cdot E \cdot I}{(2 \cdot L)^2} = \frac{2.47 \cdot E \cdot I}{L^2}$. This confirms a well-known mantra in structural analysis and the finite element method: The assumption of a displaced shape that is not entirely correct makes the structure too stiff. Here, that manifests in a higher approximate buckling load.

Aside note: Variational Formulation

In the document on energy methods, posted elsewhere on this website, the following expression are available for internal strain energy, U, and potential energy in loads, H:

Strain energy in spring: $U = \frac{1}{2} k_\theta \theta^2$

Potential energy is lateral load: $H = -F \cdot \Delta$

Potential energy in axial load, from Green's strain: $H = -P \cdot L \cdot \frac{\theta^2}{2}$

After substituting the kinematic compatibility relationship $\theta = \frac{\Delta}{L}$, the total potential energy reads:

$$\Pi = U + H = \frac{1}{2 \cdot L^2} k_\theta \Delta^2 - F \cdot \Delta - \frac{P}{2 \cdot L} \cdot \Delta^2$$

According to the principle of minimum potential energy, the variation of Π must be zero:

$$\delta\Pi = \frac{k_\theta}{L^2} \cdot \Delta \cdot \delta\Delta - F \cdot \delta\Delta - \frac{P}{L} \cdot \Delta \cdot \delta\Delta = 0$$

Rearranging yields:

$$\left(\frac{k_\theta}{L^2} \cdot \Delta - F - \frac{P}{L} \cdot \Delta \right) \delta\Delta = 0$$

For arbitrary variations, i.e., for arbitrary $\delta\Delta$, the parenthesis must be zero in order for that equation to hold true. That means, confirming the result in the previous section:

$$F = \left(\frac{k_\theta}{L^2} - \frac{P}{L} \right) \cdot \Delta$$

Inelastic Material with P-delta

In the following, the axial force, P, is expressed as a fraction of the buckling load:

$$P = \lambda P_{cr} = \lambda \frac{k_\theta}{L}$$

Prior to yielding, the governing F-Δ relationship is the same as above. After yielding the material law is

$$M = \alpha k_\theta \theta$$

where α is the strain hardening parameter that essentially represents the "second-slope stiffness," in the sense that the stiffness after yielding changes from k_θ to αk_θ. That means the governing F-Δ relationship after yielding, and including the P-delta effect, reads

$$F = \frac{M}{L} - \frac{P}{L}\Delta = \frac{M_y + \alpha\, k_\theta\, (\theta - \theta_y)}{L} - \frac{\lambda\, P_{cr}}{L}\Delta =$$

$$= \frac{M_y + \alpha\, k_\theta \left(\frac{\Delta}{L} - \frac{M_y}{k_\theta}\right)}{L} - \frac{\lambda\, k_\theta}{L^2}\Delta =$$

$$= \frac{M_y}{L}(1-\alpha) + \frac{k_\theta}{L^2}(\alpha - \lambda)\,\Delta$$

Plotting

Values:

```
E = 10 000;
fy = 40;
b = 100;
h = 200;
L = 5000;
I = b h^3 / 12;
kθ = 3 E I / L;
λ = {0.0, 0.02, 0.1, 0.5};
α = 0.02;
My = fy I / (h/2);
```

Expression from above:

```
F = If[Δ < My/kθ L, (kθ/L^2 - λ kθ/L^2) Δ, My/L (1 - α) + kθ/L^2 (α - λ) Δ];
```

Notice how collapse occurs if $\lambda > \alpha$; namely, if the fraction of the buckling load exceeds the strain hardening percentage:

```
Plot[F, {△, 0, 600}]
```

which yields: