Hjelmstad's Nonlinear Beam

This example appears in Chapter 11 of the superb textbook by Hjelmstad (1997). In Chapter 12 he extends it to numerical analysis with the Newton-Raphson method. In the following, portions of that example are explained. Any errors and obscurity that may appear in this document are due to my own misrepresentation of Hjelmstad's beautiful work. On the same note, some sign conventions for stress resultants and the shear angle are different in this document, following other documents posted on this website. The starting point for the derivations is equilibrium. Documents posted on this website, related to energy methods and also various boundary value problems, give an introduction to establishing the weak form with equilibrium as a starting point. As an example, consider Euler-Bernoulli beam theory, in which equilibrium in the *z*-direction reads:

$$-\frac{dV}{dx}+q_z=0 \tag{1}$$

and counter-clockwise moment equilibrium reads

$$-\frac{dM}{dx}-V=0 \tag{2}$$

Weighting and integration of those equations, with δw (acts in the *z*-direction) and $\delta\theta$ (acts counter-clockwise), respectively, while introducing the notation $\bullet/dx \equiv \bullet'$ gives

$$\int_0^L (-V'+q_z)\cdot\delta w\,dx=0 \tag{3}$$

and

$$\int_0^L (-M'-V)\cdot\delta\theta\,dx=0 \tag{4}$$

Integration by parts, in order to avoid derivatives of the stress resultants, yields, after cancelling boundary terms:

$$\int_0^L (V\cdot\delta w'+q_z\cdot\delta w)\,dx=0 \tag{5}$$

and

$$\int_0^L (M\cdot\delta\theta'-V\cdot\delta\theta)\,dx=0 \tag{6}$$

The terms in Eqs. (5) and (6) are now sorted into external and internal virtual work. The external virtual work is recognized from documents on this website where the load vector for beam elements is developed:

$$\delta W_{\text{ext}} = \int_0^L q_z \cdot \delta w \, dx \tag{7}$$

The internal virtual work contains terms from both Eq. (5) and Eq. (6):

$$\delta W_{\text{int}} = \int_0^L (M \cdot \delta\theta' - V \cdot \delta\theta + V \cdot \delta w') \, dx \tag{8}$$

V cancels against V. By neglecting shear deformation, we have that $\delta\theta=\delta w'$, and Eq. (7) simplifies to

$$\delta W_{\text{int}} = \int_0^L (M \cdot \delta w'') \, dx \tag{9}$$

Finally, introducing $M=EI\,w''$, i.e., material law with kinematic compatibility built into it, we arrive at the well-known internal virtual work for beam bending:

$$\delta W_{\text{int}} = \int_0^L (EI \cdot w'' \cdot \delta w'') \, dx \tag{10}$$

A similar procedure is observed in the following.

Equilibrium

Consider the infinitesimally small portion of a beam shown in Figure 1. It is subjected to distributed load, q_z, in the z-direction, distributed load, q_x, in the x-direction, as well as distributed counter-clockwise moment, m. Considering the left-hand side of Figure 1, equilibrium in the x-direction yields

$$-N + N + dN + q_x \cdot dx = 0 \Longrightarrow N' + q_x = 0 \tag{11}$$

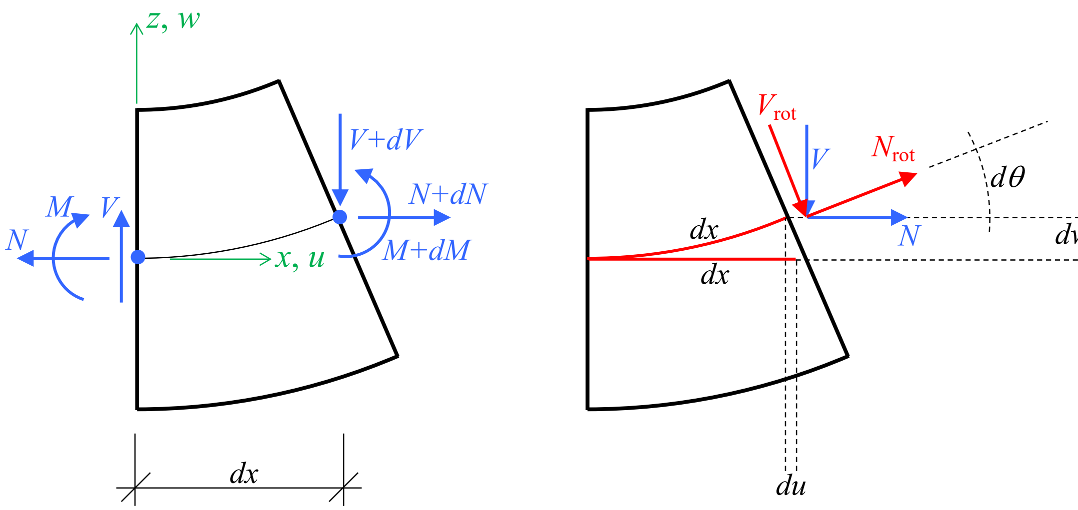

Figure 1: Portion of a beam with large displacements.

Equilibrium in the z-direction yields

$$V - V - dV + q_z \cdot dx = 0 \Longrightarrow -V' + q_z = 0 \tag{12}$$

Moment equilibrium, with counter-clockwise as positive, about the right-hand side edge of the infinitesimal beam element yields

$$-M + M + dM - V(dx + du) - N\,dw + m\,dx = 0 \tag{13}$$

where loading terms containing dx^2 are neglected. Cancelling M against M and dividing through by dx yields

$$M' - V(1 + u') - N\,w' + m = 0 \tag{14}$$

For future reference, before leaving the equilibrium section, in order to account for large deformations, N and V are related to the rotated stress resultants N_{rot} and V_{rot}, shown on the right-hand side of Figure 1:

$$N = N_{\text{rot}} \cdot \cos(\theta) + V_{\text{rot}} \cdot \sin(\theta) \tag{15}$$

$$V = V_{\text{rot}} \cdot \cos(\theta) - N_{\text{rot}} \cdot \sin(\theta) \tag{16}$$

Virtual Work

Eqs. (11), (12), and (14) are the governing equilibrium equations in the x, z, and rotational directions. They are now weighted by δu, δw, and $\delta\theta$, respectively, and then integrated from 0 to L. First, the x-direction:

$$\int_0^L (N' + q_x)\,\delta u\,dx = 0 \tag{17}$$

Next, the z-direction:

$$\int_0^L (-V' + q_z)\,\delta w\,dx = 0 \tag{18}$$

Finally, the counter-clockwise rotational direction:

$$\int_0^L (M' - V(1 + u') - N\,w' + m)\,\delta\theta\,dx = 0 \tag{19}$$

Next, the weight functions are multiplied into the parentheses, and integration by parts is carried out in order to avoid derivatives on the stress resultants. Boundary terms cancel:

$$\int_0^L (-N\delta u' + q_x\delta u)\,dx = 0 \tag{20}$$

$$\int_0^L (V\delta w' + q_z\delta w)\,dx = 0 \tag{21}$$

$$\int_0^L (-M\delta\theta' - V(1+u')\delta\theta - N\, w'\delta\theta + m\,\delta\theta)\, dx = 0 \tag{22}$$

All terms in Eqs. (20), (21), and (22) are now sorted as either external or internal virtual work:

$$\delta W_{\text{ext}} = \int_0^L (q_x\delta u + q_z\delta w + m\,\delta\theta)\, dx \tag{23}$$

$$\delta W_{\text{int}} = \int_0^L (-N\delta u' + V\delta w' - M\delta\theta' - V(1+u')\delta\theta - N\, w'\delta\theta)\, dx \tag{24}$$

Next, Eq. (24) is reorganized by stress resultant:

$$\delta W_{\text{int}} = \int_0^L (-N(\delta u' + w'\delta\theta) + V(\delta w' - (1+u')\delta\theta) - M\delta\theta')\, dx \tag{25}$$

This is a good time to insert the equilibrium equations (15) and (16) into Eq. (25) in order to express the internal virtual work in terms of the rotated stress resultants:

$$\delta W_{\text{int}} = \int_0^L \begin{pmatrix} -(N_{\text{rot}}\cdot\cos(\theta) + V_{\text{rot}}\cdot\sin(\theta))(\delta u' + w'\delta\theta) \\ +(V_{\text{rot}}\cdot\cos(\theta) - N_{\text{rot}}\cdot\sin(\theta))(\delta w' - (1+u')\delta\theta) \\ -M\delta\theta' \end{pmatrix} dx \tag{26}$$

We reorganize again by stress resultant to obtain

$$\delta W_{\text{int}} = \int_0^L \begin{pmatrix} N_{\text{rot}}(-\delta u'\cos(\theta) - w'\delta\theta\cos(\theta) - \delta w'\sin(\theta) + (1+u')\delta\theta\sin(\theta)) \\ +V_{\text{rot}}(-\delta u'\sin(\theta) - w'\delta\theta\sin(\theta) + \delta w'\cos(\theta) - (1+u')\delta\theta\cos(\theta)) \\ +M(\delta\theta') \end{pmatrix} dx \tag{27}$$

The parentheses that the stress resultants multiply with are by definition the corresponding virtual strains, i.e., virtual axial strain $\delta\varepsilon$, shear angle $\delta\gamma$, and curvature $\delta\kappa$. In other words, the virtual axial strain is

$$\delta\varepsilon = -\delta u'\cos(\theta) - w'\cos(\theta)\delta\theta - \delta w'\sin(\theta) + (1+u')\sin(\theta)\delta\theta \tag{28}$$

Similarly, from Eq. (27), the virtual shear angle is

$$\delta\gamma = -\delta u'\sin(\theta) - w'\sin(\theta)\,\delta\theta + \delta w'\cos(\theta) - (1+u')\cos(\theta)\,\delta\theta \tag{29}$$

and the virtual curvature is simply $\delta\theta'$.

Real Strains

In order to obtain the corresponding real strains, "anti-variation" from variational calculus is carried out on Eqs. (28) and (29). This is nicely explained in Hjelmstad's book via Vainberg's theorem. As an introduction, we here arbitrarily consider the functional $w'\cdot\cos(\theta)$. Recognizing w and θ as independent functions, taking the variation yields $\delta w'\cdot\cos(\theta) - w'\cdot\sin(\theta)\cdot\delta\theta$. What we want in Eqs. (28) and (29) is to go the other way, i.e., anti-variation. That becomes a puzzle of symmetry, as explained by Hjelmstad, meaning that we must recognize that the anti-variation of $\delta w' \cdot \cos(\theta) - w' \cdot \sin(\theta) \cdot \delta\theta$ is $w' \cdot \cos(\theta)$. That is a bookkeeping issue. To that end, the terms in Eq. (28) are examined in Figure 2 and the terms in Eq. (29) are examined in Figure 3. The result from the anti-variation observed in those figures is

$$\varepsilon = -w' \sin(\theta) - (1 + u') \cos(\theta) + 1 \tag{30}$$

$$\gamma = w' \cos(\theta) - (1 + u') \sin(\theta) \tag{31}$$

where the "plus one" in Eq. (30) is introduced in order to observe zero strain in the presence of zero deformation.

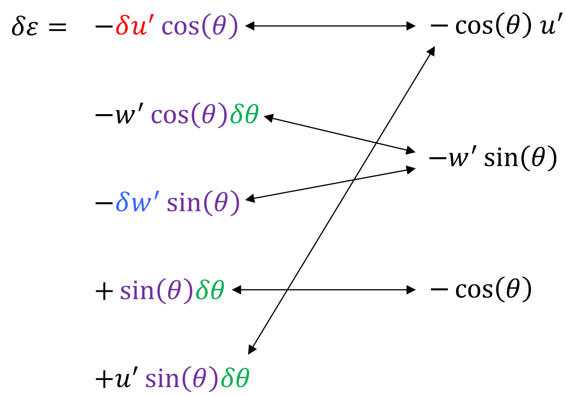

Figure 2: Examining terms to conduct anti-variation for axial strain.

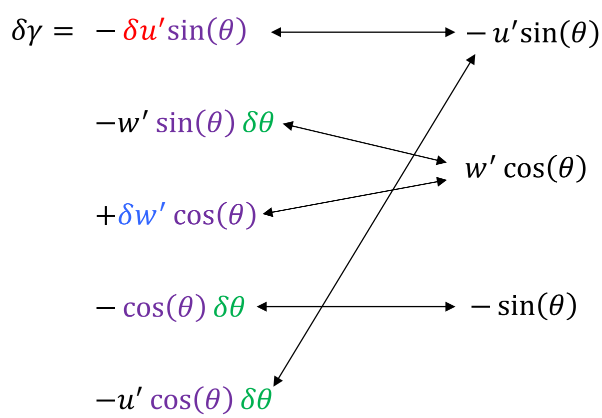

Figure 3: Examining terms to conduct anti-variation for shear strain.

Material Law

The relationship between stress resultants and corresponding deformations are here referred to as the material law, although they also contain kinematic compatibility equations at the element level. The relationship between axial force and axial strain is

$$N = EA \cdot \varepsilon \tag{32}$$

The relationship between shear force and shear angle is (see the document on Timonshenko beams)

$$V = GA_v \cdot \gamma \tag{33}$$

Finally, the relationship between bending moment and curvature is

$$M = EI \cdot \kappa \tag{34}$$

where $\kappa = \theta'$.

Simplifications, Option 1: Euler's Elastica

Before utilizing the full formulation established above in nonlinear analysis, a couple of simplifications are explored. One is to neglect axial deformation. Interestingly, this restraint can be enforced by considering Eq. (30). By setting $w'=\sin(\theta)$ and $1+u'=\cos(\theta)$ that equation takes the form $\varepsilon = -\sin^2(\theta) - \cos^2(\theta) + 1$, which equals zero axial strain. As a result, Eq. (14) turns into

$$M' - V \cdot \cos(\theta) - N \cdot \sin(\theta) + m = 0 \tag{35}$$

This is Euler's elastica theory, and a further simplification is introduced by considering a cantilevered column without shear force, simply with an axial force, P, at the top. This means that Eq. (35) reads, once $M=EI\cdot\theta'$ is introduced:

$$EI \cdot \theta'' - P \cdot \sin(\theta) = 0 \tag{36}$$

with boundary conditions $\theta(0)=0$ and $\theta'(L)=0$. One manner in which to employ Eq. (36) is to weight & integrate it in order to obtain the "weighted residual form" of the boundary value problem:

$$\int_0^L (EI \cdot \theta'' - P \cdot \sin(\theta))\, \delta\theta \, dx = 0 \tag{37}$$

followed by integration by parts, as shown in other documents on this website, in order to obtain the weak form:

$$\int_0^L (EI \cdot \theta' \cdot \delta\theta' - P \cdot \sin(\theta) \cdot \delta\theta) \, dx = 0 \tag{38}$$

Anti-variation from variational calculus yields

$$\int_0^L \left(\frac{1}{2} \cdot EI \cdot (\theta')^2 + P \cdot \cos(\theta)\right) dx = 0 \tag{39}$$

which is the energy form with the second term comparable to Green's strain, with two caveats: 1) Green's strain is an approximation; 2) The potential energy in P is here measured with starting value PL instead of zero, as it is when the energy is expressed as $P\Delta_{vert}$.

Simplifications, Option 2: Linearized Buckling Theory

Another way to simplify the previously established theory is to assume small θ, which means that $\sin(\theta)\approx\theta$ and $\cos(\theta)\approx 1$. It is also assumed that $w'\approx\theta$ and $u'\approx 0$. Consider the previously established equilibrium equations and differentiate Eq. (14) once, followed by substitution of Eq. (12):

$$M'' - (N\,w')' - q_z + m' = 0 \tag{40}$$

Substitution of $M=EIw''$ together with the assumption that EI and N are constant gives

$$EI\,w'''' - N\,w'' - q_z + m' = 0 \tag{41}$$

Weighting & integration yields

$$\int_0^L (EI\,w'''' - N\,w'' - q_z + m')\,\delta w\,dx = 0 \tag{42}$$

Integration by parts yields the weak form:

$$\int_0^L (EI\,w''\delta w'' - N\,w'\delta w' - q_z\delta w + m\,\delta w')\,dx = 0 \tag{43}$$

Anti-variation yields the variational form, i.e., expressed in terms of energy:

$$\int_0^L \left(\frac{1}{2}EI\,(w'')^2 - N\frac{1}{2}(w')^2 - q_z\,w + m\,w'\right)dx = 0 \tag{44}$$

In comparison with Eq. (39), notice how the potential energy in the axial force is now expressed in terms of Green's strain; see documents on energy methods and the truss with geometric nonlinearity. Also, the potential energy in the axial force is now measured with starting value zero, i.e., that potential energy is expressed in terms of $P\Delta_{\text{vert}}$.

No Simplifications: Full Nonlinear Analysis

The virtual work formulation established earlier represents $\delta W_{\text{int}}=\delta W_{\text{ext}}$ with δW_{ext} given in Eq. (23) and δW_{int} given in Eq. (27) so that the generic form is

$$\int_0^L (EA\,\varepsilon\,\delta\varepsilon + GA_v\,\gamma\,\delta\gamma + EI\,\kappa\,\delta\kappa)\,dx = \int_0^L (q_x\delta u + q_z\delta w + m\,\delta\theta)\,dx \tag{45}$$

with $\delta\varepsilon$ and $\delta\gamma$ defined in Eqs. (28) and (29), and ε and κ defined in Eqs. (30) and (31). Following the last chapter in Hjelmstad's book, the objective is now to employ this full formulation in a nonlinear analysis with the Newton-Raphson algorithm. To understand how this is done, concepts from the linear and nonlinear finite element method explained on the Finite Elements page of this website are employed:

1. Establish the weak form of the boundary value problem; this is already done above and summarized in Eq. (45)

2. Collect the unknown field functions in the vector $\tilde{\mathbf{u}}(x) \equiv \{u(x),\ w(x),\ \theta(x)\}$ and express the weak form as the functional $G(\lambda,\ \tilde{\mathbf{u}},\ \delta\tilde{\mathbf{u}}) = \delta W_{\text{int}} - \delta W_{\text{ext}}$, where λ is the load factor
3. Discretize the problem; in the generic finite element method this is done by $\tilde{\mathbf{u}}(x)=\mathbf{N}(x)\mathbf{u}$ and here by $\tilde{\mathbf{u}}(x)=\mathbf{h}(x)\mathbf{a}$, where **a** are unknown constants, often called generalized degrees of freedom, but not actual displacement degrees of freedom; this is known as the Ritz method after work by Walter Ritz published in 1908
4. Use the same approximation for virtual displacements as for the real displacements; this is known as the Galerkin approach
5. Taylor linearize $G \approx G(\mathbf{a}_i)+\nabla G(\mathbf{a}_i)\cdot\Delta\mathbf{a}$ and formulate each Newton-Raphson iteration as $\mathbf{a}_{i+1} = \mathbf{a}_i + \Delta\mathbf{a}$ with $\Delta\mathbf{a}$ from solving $G(\mathbf{a}_i)+\nabla G(\mathbf{a}_i)\cdot\Delta\mathbf{a}=0$

While shape functions, $\mathbf{N}(x)$, are spelled out for various finite elements elsewhere on this website, the "basis functions" $\mathbf{h}(x)$ require further attention here.

Working on this document in the spring of 2024…

References

Hjelmstad, K. D. (1997). "Fundamentals of Structural Mechanics." Prentice Hall. (This is the first edition of the book; a second edition is available.)