Equilibrium

Equilibrium equations complement material law and kinematic compatibility in order to complete the governing equations for a boundary value problem. However, the equilibrium side of the problem has unique features for structural problems. First, we often employ the principle of virtual displacements in order to obtain solutions. In that case, that principle actually takes care of equilibrium, albeit in an average sense. Furthermore, for structural problems we do not directly link externally applied loads with the internal stresses in the material. Rather, we work with stress resultants, such as bending moment. That means we link external loads with the stress resultants, and then the stress resultants with stress. That said, however, it is important to establish two sets of *generic* equilibrium equations. One is equilibrium of a tetrahedron subjected to a surface stress "traction." That is what Augustin Cauchy did in 1822 when he introduced the concept of stress. That exercise, involving surface tractions, is carried out in the document on stress transformations, posted near this one. The other important set of generic equilibrium equations for a material particle address body force, rather than surface tractions. Even in the absence of a body force, these considerations give equations that all stress states must satisfy, in order to be in equilibrium. Figure 1 shows the stresses acting on an infinitesimally small material volume.

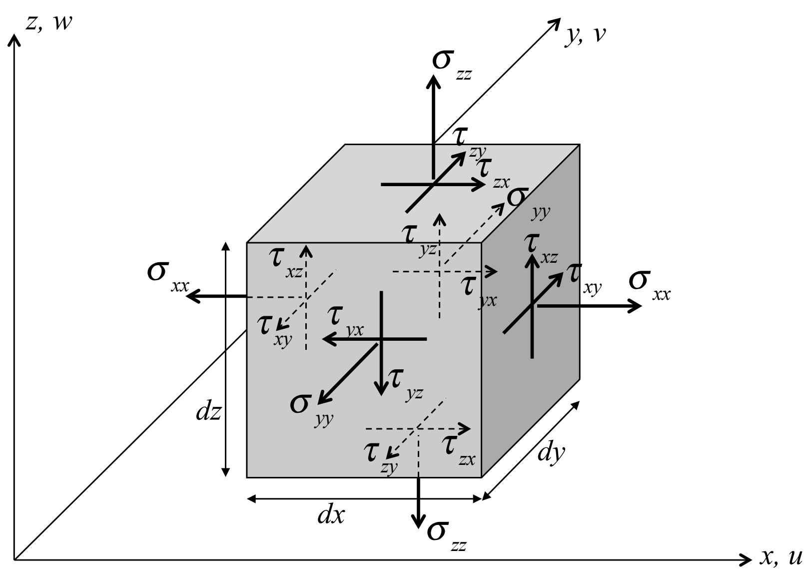

Figure 1: Coordinate stresses.

The figure is a bit hard to read because it includes the changes in stress-values from one end of the cube to the opposite side. For example, the axial stress σ_{xx} changes by $d\sigma_{xx}$ from one side to the other. The volume forces are not included in the figure but are denoted f_x, f_y, and f_z, with the index indicating the direction of the force. Equilibrium in the x-direction yields

$$d\sigma_{xx} \cdot dy \cdot dz + d\sigma_{yx} \cdot dx \cdot dz + d\sigma_{zx} \cdot dx \cdot dy + f_x \cdot dx \cdot dy \cdot dz = 0 \tag{1}$$

Dividing through by (*dx dy dz*) yields:

$$\frac{d\sigma_{xx}}{dx}+\frac{d\sigma_{yx}}{dy}+\frac{d\sigma_{zx}}{dz}+f_x=0 \tag{2}$$

Repeating the exercise in all three axis-directions produces the equilibrium equations that all material particles that are in equilibrium must satisfy, here expressed in index notation:

$$\sigma_{ij,i}+f_j=0 \tag{3}$$

Moment equilibrium of the infinitesimal cube yields the symmetry of the stress tensor:

$$\sigma_{ij}=\sigma_{ji} \tag{4}$$

References

Timoshenko, S., and Goodier, J. N. (1969). *Theory of elasticity*. McGraw-Hill.