Rayleigh Damping Derivatives

In the documents on the direct differentiation method for linear and nonlinear dynamics, the derivative of the damping matrix, $\partial\mathbf{C}/\partial\theta$, appears but is not further developed. This document addresses the case of Rayleigh damping, which means that the damping matrix is

$$\mathbf{C} = c_M\mathbf{M} + c_K\mathbf{K} \tag{1}$$

where c_M and c_K are scalars, while $\mathbf{M}$ and $\mathbf{K}$ are the mass and stiffness matrix, respectively. Many options exist within the framework set forth by Eq. (1). The coefficients c_M and c_K can be given as numbers by the analyst, or they are calculated as part of the analysis from target damping specified at two natural frequencies. In the latter case, the value of c_M and c_K depends on $\mathbf{M}$ and $\mathbf{K}$, because those matrices enter the eigenvalue problem. As a result, c_M and c_K depend on θ, if that parameter enters $\mathbf{M}$ and $\mathbf{K}$. Furthermore, in nonlinear dynamics, $\mathbf{K}$ can be the initial stiffness matrix, the committed tangent stiffness at the previously converged equilibrium state, or the current tangent stiffness updated at every Newton-Raphson iteration. In the first of those cases, $\partial\mathbf{K}/\partial\theta$ is calculated in a straightforward manner. In the latter two cases, $\mathbf{K}$ depends on the displacement, $\mathbf{u}$, meaning that $\mathbf{K}$ may depend on θ both implicitly via $\mathbf{u}$ and explicitly via the algorithm that calculates $\mathbf{K}$. That complicate matters, also because c_M and c_K depends on $\mathbf{K}$. Ordered by increasing complexity, the following options are considered in the subsequent sections of this document:

- Coefficients c_M and c_K input as numbers with $\mathbf{K}$ *not* depending on $\mathbf{u}$
- Coefficients c_M and c_K input as numbers with $\mathbf{K}$ depending on $\mathbf{u}$
- Coefficients c_M and c_K calculated by the program with $\mathbf{K}$ *not* depending on $\mathbf{u}$
- Coefficients c_M and c_K calculated by the program and $\mathbf{K}$ depending on $\mathbf{u}$

One reference for this document is the paper entitled "Exact Sensitivity of Nonlinear Dynamic Response with Modal and Rayleigh Damping Formulated with the Tangent Stiffness" that I recently published in the ASCE Journal of Structural Engineering."

Given Coefficients, Initial Stiffness

In the first and simplest case listed above, the sought derivative is obtained from the product rule of differentiation:

$$\frac{\partial\mathbf{C}}{\partial\theta} = \frac{\partial c_M}{\partial\theta}\mathbf{M} + c_M\frac{\partial\mathbf{M}}{\partial\theta} + \frac{\partial c_K}{\partial\theta}\mathbf{K} + c_K\frac{\partial\mathbf{K}}{\partial\theta} \tag{2}$$

where the value of $\partial c_M/\partial\theta$ and $\partial c_K/\partial\theta$ is zero or unity, depending on whether θ represents c_M or c_K. Also, $\partial\mathbf{M}/\partial\theta$ and $\partial\mathbf{K}/\partial\theta$ are straightforward and identical to those already appearing in the right-hand side of the linear system of equations for $\partial\mathbf{u}/\partial\theta$ in linear dynamics.

Given Coefficients, Tangent Stiffness

Now consider the case when c_M and c_K are still given by the analyst as numbers, while $\mathbf{K}$ is the tangent stiffness and therefor depends on $\mathbf{u}$. Now, the stiffness matrix is implicitly dependent upon θ via the displacement response and also potentially explicitly dependent upon θ via the calculations and assembly of $\mathbf{K}$. As is done for the internal force in nonlinear static and dynamic analysis, that implicit and explicit dependence is expressed as

$$\mathbf{C} = c_M \mathbf{M} + c_K \mathbf{K}(\mathbf{u}(\theta), \theta) \tag{3}$$

Differentiation yields

$$\frac{\partial \mathbf{C}}{\partial \theta} = \frac{\partial c_M}{\partial \theta}\mathbf{M} + c_M \frac{\partial \mathbf{M}}{\partial \theta} + \frac{\partial c_K}{\partial \theta}\mathbf{K} + c_K \left(\frac{\partial \mathbf{K}}{\partial \mathbf{u}_{n+1}} \frac{\partial \mathbf{u}_{n+1}}{\partial \theta} + \left. \frac{\partial \mathbf{K}(\mathbf{u}_{n+1})}{\partial \theta} \right|_{\mathbf{u}\text{ fixed}} \right) \tag{4}$$

Interestingly, the third-order tensor $\partial\mathbf{K}/\partial\mathbf{u}$ appears and, importantly, the sought response sensitivity $\partial\mathbf{u}_{n+1}/\partial\theta$ also appears in Eq. (4). To address that issue, the first term in the parenthesis in Eq. (4) is first substituted into the correct term in the right-hand side of the system of equations for $\partial\mathbf{u}_{n+1}/\partial\theta$, which yields

$$(\cdots)\frac{\partial \mathbf{u}_{n+1}}{\partial \theta} = \cdots - c_K \frac{\partial \mathbf{K}}{\partial \mathbf{u}_{n+1}} \frac{\partial \mathbf{u}_{n+1}}{\partial \theta} (a_5 \mathbf{u}_{n+1} + a_6 \mathbf{u}_n + a_7 \dot{\mathbf{u}}_n + a_8 \ddot{\mathbf{u}}_n) \tag{5}$$

In index notation, with summation implied over repeated indices the, right-hand side of Eq. (5) reads, after some *ad hoc* relabelling of symbols:

$$(\cdots)\frac{\partial \mathbf{u}_{n+1}}{\partial \theta} = \cdots - c_K \cdot dKu_{imj} \cdot du_j \cdot au_m \tag{6}$$

Notice that the parenthesis $(a_5 \mathbf{u}_{n+1} + a_6 \mathbf{u}_n + a_7 \dot{\mathbf{u}}_n + a_8 \ddot{\mathbf{u}}_n)$ is contracted with the middle index of the tensor $\partial\mathbf{K}/\partial\mathbf{u}$. That leads to the rewrite

$$\begin{aligned}(\cdots)\frac{\partial \mathbf{u}_{n+1}}{\partial \theta} &= \cdots - c_K \big(dKu_{imj} \cdot au_m\big) du_j \\ &= \cdots - \left(c_K \frac{\partial \boldsymbol{K}}{\partial \mathbf{u}_{n+1}} \circ (a_5 \mathbf{u}_{n+1} + a_6 \mathbf{u}_n + a_7 \dot{\mathbf{u}}_n + a_8 \ddot{\mathbf{u}}_n) \right) \frac{\partial \mathbf{u}_{n+1}}{\partial \theta}\end{aligned} \tag{7}$$

where $\circ$ means contraction on the middle index, as mentioned, which facilitates the collection of all terms with $\partial\mathbf{u}_{n+1}/\partial\theta$ on the left-hand side:

$$\left(\cdots + c_K \frac{\partial \boldsymbol{K}}{\partial \mathbf{u}_{n+1}} \circ (a_5 \mathbf{u}_{n+1} + a_6 \mathbf{u}_n + a_7 \dot{\mathbf{u}}_n + a_8 \ddot{\mathbf{u}}_n) \right) \frac{\partial \mathbf{u}_{n+1}}{\partial \theta} = \cdots \tag{8}$$

In conclusion, the use of the current tangent stiffness in Rayleigh damping leads to an amended effective stiffness in the linear system of equations for the response sensitivities.

Calculated Coefficients, Initial Stiffness

Now consider the case where the coefficients c_M and c_K are specified by a target damping ratio at two natural frequencies in the following manner:

$$c_M = \omega_1 \cdot \omega_2 \cdot \frac{2 \cdot \zeta_{\text{target}}}{\omega_1 + \omega_2} \tag{9}$$

and

$$c_K = \frac{2 \cdot \zeta_{\text{target}}}{\omega_1 + \omega_2} \tag{10}$$

In this case, the calculation of $\partial c_M/\partial \theta$ and $\partial c_K/\partial \theta$ require the differentiation of the eigenvalue problem from which the natural frequencies are determined. That is because the parameter θ may affect the natural frequencies, which in turn determine c_M and c_K. The direct differentiation method for calculating eigenvalue derivatives is addressed in another document posted near this one. Suppose the eigenvalues are denoted by the symbol γ, with natural frequencies then being the square root of γ. According to Eqs. (9) and (10), the sought derivatives are

$$\frac{\partial c_M}{\partial \theta} = \frac{\partial \omega_1}{\partial \theta} \cdot \omega_2 \cdot \frac{2 \cdot \zeta_{\text{target}}}{\omega_1 + \omega_2} + \omega_1 \cdot \frac{\partial \omega_2}{\partial \theta} \cdot \frac{2 \cdot \zeta_{\text{target}}}{\omega_1 + \omega_2} - \omega_1 \cdot \omega_2 \cdot \frac{2 \cdot \zeta_{\text{target}}}{(\omega_1 + \omega_2)^2} \cdot \left(\frac{\partial \omega_1}{\partial \theta} + \frac{\partial \omega_2}{\partial \theta} \right) \tag{11}$$

and

$$\frac{\partial c_K}{\partial \theta} = -\frac{2 \cdot \zeta_{\text{target}}}{(\omega_1 + \omega_2)^2} \cdot \left(\frac{\partial \omega_1}{\partial \theta} + \frac{\partial \omega_2}{\partial \theta} \right) \tag{12}$$

where

$$\frac{\partial \omega}{\partial \theta} = \frac{1}{2} \cdot \frac{1}{\sqrt{\gamma}} \cdot \frac{\partial \gamma}{\partial \theta} \tag{13}$$

Calculated Coefficients, Tangent Stiffness

Next, consider the most complex case listed in the introduction. Accounting for both the implicit and explicit dependence of c_M and c_K on θ, the sought derivatives read

$$\frac{\partial c_M}{\partial \theta} = \left(\frac{\partial c_M}{\partial \omega_1} \cdot \frac{\partial \omega_1}{\partial \gamma_1} \cdot \frac{\partial \gamma_1}{\partial \mathbf{K}} + \frac{\partial c_M}{\partial \omega_2} \cdot \frac{\partial \omega_2}{\partial \gamma_2} \cdot \frac{\partial \gamma_2}{\partial \mathbf{K}} \right) \cdot \frac{\partial \mathbf{K}}{\partial \mathbf{u}} \cdot \frac{\partial \mathbf{u}}{\partial \theta} + \left. \frac{\partial c_M}{\partial \theta} \right|_{\mathbf{u}\text{ fixed}} \tag{14}$$

and

$$\frac{\partial c_K}{\partial \theta} = \left(\frac{\partial c_K}{\partial \omega_1} \cdot \frac{\partial \omega_1}{\partial \gamma_1} \cdot \frac{\partial \gamma_1}{\partial \mathbf{K}} + \frac{\partial c_K}{\partial \omega_2} \cdot \frac{\partial \omega_2}{\partial \gamma_2} \cdot \frac{\partial \gamma_2}{\partial \mathbf{K}} \right) \cdot \frac{\partial \mathbf{K}}{\partial \mathbf{u}} \cdot \frac{\partial \mathbf{u}}{\partial \theta} + \left. \frac{\partial c_K}{\partial \theta} \right|_{\mathbf{u}\text{ fixed}} \tag{15}$$

First addressing the unconditional derivatives, the first factor is, for c_M

$$\frac{\partial c_M}{\partial \omega_1} = \omega_2 \cdot \frac{2 \cdot \zeta_{\text{target}}}{\omega_1 + \omega_2} - \omega_1 \cdot \omega_2 \cdot \frac{2 \cdot \zeta_{\text{target}}}{(\omega_1 + \omega_2)^2} \tag{16}$$

and

$$\frac{\partial c_M}{\partial \omega_2} = \omega_1 \cdot \frac{2 \cdot \zeta_{\text{target}}}{\omega_1 + \omega_2} - \omega_1 \cdot \omega_2 \cdot \frac{2 \cdot \zeta_{\text{target}}}{(\omega_1 + \omega_2)^2} \tag{17}$$

Similarly, for c_K,

$$\frac{\partial c_K}{\partial \omega_1} = -\frac{2 \cdot \zeta_{\text{target}}}{(\omega_1 + \omega_2)^2} \tag{18}$$

and

$$\frac{\partial c_K}{\partial \omega_2} = -\frac{2 \cdot \zeta_{\text{target}}}{(\omega_1 + \omega_2)^2} \tag{19}$$

The next factor is

$$\frac{\partial \omega_1}{\partial \gamma_1} = \frac{1}{2\sqrt{\gamma}} \tag{20}$$

The third factor, $d\gamma/d\mathbf{K}$, is addressed in the document on eigenvalue derivatives. The conditional derivatives are the same as those given in Eqs. (11), (12), and (13):

$$\left.\frac{\partial c_M}{\partial \theta}\right|_{\mathbf{u}\text{ fixed}} = \left(\frac{\partial c_M}{\partial \omega_1} \cdot \frac{\partial \omega_1}{\partial \gamma_1} \cdot \frac{\partial \gamma_1}{\partial \theta} + \frac{\partial c_M}{\partial \omega_2} \cdot \frac{\partial \omega_2}{\partial \gamma_2} \cdot \frac{\partial \gamma_2}{\partial \theta}\right) \tag{21}$$

and

$$\left.\frac{\partial c_K}{\partial \theta}\right|_{\mathbf{u}\text{ fixed}} = \left(\frac{\partial c_K}{\partial \omega_1} \cdot \frac{\partial \omega_1}{\partial \gamma_1} \cdot \frac{\partial \gamma_1}{\partial \theta} + \frac{\partial c_K}{\partial \omega_2} \cdot \frac{\partial \omega_2}{\partial \gamma_2} \cdot \frac{\partial \gamma_2}{\partial \theta}\right) \tag{22}$$

Now reconsider the unconditional derivatives, i.e., the first term in Eqs. (14) and (15). Similar to the derivations behind Eq. (8), the right-hand side of the system of equations for the response sensitivities, $\partial \mathbf{u}_{n+1}/\partial \theta$, is addressed. The relevant terms are

$$-\frac{\partial c_M}{\partial \theta}\mathbf{M}(a_5\mathbf{u}_{n+1} + a_6\mathbf{u}_n + a_7\dot{\mathbf{u}}_n + a_8\ddot{\mathbf{u}}_n) - \frac{\partial c_K}{\partial \theta}\mathbf{K}(a_5\mathbf{u}_{n+1} + a_6\mathbf{u}_n + a_7\dot{\mathbf{u}}_n + a_8\ddot{\mathbf{u}}_n) \tag{23}$$

Substitution of the first term in Eqs. (14) and (15) yields

$$
\begin{aligned}
&-\left(\frac{\partial c_M}{\partial \omega_1}\cdot\frac{\partial \omega_1}{\partial \gamma_1}\cdot\frac{\partial \gamma_1}{\partial \mathbf{K}}+\frac{\partial c_M}{\partial \omega_2}\cdot\frac{\partial \omega_2}{\partial \gamma_2}\cdot\frac{\partial \gamma_2}{\partial \mathbf{K}}\right)\cdot\frac{\partial \mathbf{K}}{\partial \mathbf{u}}\cdot\frac{\partial \mathbf{u}}{\partial \theta}\cdot\mathbf{M}\cdot\left(a_5\mathbf{u}_{n+1}+a_6\mathbf{u}_n+a_7\dot{\mathbf{u}}_n+a_8\ddot{\mathbf{u}}_n\right)\\
&-\left(\frac{\partial c_K}{\partial \omega_1}\cdot\frac{\partial \omega_1}{\partial \gamma_1}\cdot\frac{\partial \gamma_1}{\partial \mathbf{K}}+\frac{\partial c_K}{\partial \omega_2}\cdot\frac{\partial \omega_2}{\partial \gamma_2}\cdot\frac{\partial \gamma_2}{\partial \mathbf{K}}\right)\cdot\frac{\partial \mathbf{K}}{\partial \mathbf{u}}\cdot\frac{\partial \mathbf{u}}{\partial \theta}\cdot\mathbf{K}\cdot\left(a_5\mathbf{u}_{n+1}+a_6\mathbf{u}_n+a_7\dot{\mathbf{u}}_n+a_8\ddot{\mathbf{u}}_n\right)
\end{aligned}
\tag{24}
$$

As earlier in this document, the response sensitivity, $\partial\mathbf{u}/\partial\theta$, is isolated in order to amend the coefficient matrix of the linear system of equations from which it is solved. Index notation, here informally stated as $-dc_M K_{ij}\ B_{ijm}\ du_m\ M_{ln}\ a_n - dc_K K_{ij}\ B_{ijm}\ du_m\ M_{ln}\ a_n =$ $= -(dc_M K_{ij}\ B_{ijm}\ M_{ln}\ a_n)\ du_m - (dc_K K_{ij}\ B_{ijm}\ M_{ln}\ a_n)\ du_m$, suggests that the three-dimensional tensor d**K**/d**u** is never fully contracted; the last index remains free and contracts with d**u**/dθ. In addition, note that the resulting outer product of the vectors $\left(\frac{\partial c_M}{\partial \omega_1}\cdot\frac{\partial \omega_1}{\partial \gamma_1}\cdot\frac{\partial \gamma_1}{\partial \mathbf{K}}+\frac{\partial c_M}{\partial \omega_2}\cdot\frac{\partial \omega_2}{\partial \gamma_2}\cdot\frac{\partial \gamma_2}{\partial \mathbf{K}}\right)\cdot\frac{\partial \mathbf{K}}{\partial \mathbf{u}}$ and $\mathbf{M}\cdot\left(a_5\mathbf{u}_{n+1}+a_6\mathbf{u}_n+a_7\dot{\mathbf{u}}_n+a_8\ddot{\mathbf{u}}_n\right)$ is asymmetric.