Modal Damping Derivatives

A reference for the equations presented in this document is the paper entitled "Exact Sensitivity of Nonlinear Dynamic Response with Modal and Rayleigh Damping Formulated with the Tangent Stiffness" that I recently published in the ASCE Journal of Structural Engineering." Long before that, an article by Chopra & McKenna (2016) explains the advantages of modal damping over Rayleigh damping in nonlinear dynamic analysis. The modal damping matrix is

$$\mathbf{C} = \mathbf{M}\left(\sum_{j=1}^{N} \frac{2\zeta_j \omega_j}{m_j} \boldsymbol{\phi}_j \boldsymbol{\phi}_j^{\mathrm{T}}\right)\mathbf{M} \tag{1}$$

where $\mathbf{M}$ is the mass matrix, ζ_j is the damping ratio at mode j, ω_j is the natural frequency of vibration in mode j, $\boldsymbol{\phi}_j$ is the corresponding mode shape, and m_j is the modal mass, i.e., $\boldsymbol{\phi}_j^{\mathrm{T}}\mathbf{M}\boldsymbol{\phi}_j$. The objective in this document is to calculate the derivative $\partial\mathbf{C}/\partial\theta$, where θ is an input parameter to the finite element model. The product rule of differentiation yields

$$\begin{aligned}
\frac{d\mathbf{C}}{d\theta} \\
= \frac{d\mathbf{M}}{d\theta}\left(\sum_{j=1}^{N} \frac{2\zeta_j \omega_j}{m_j} \boldsymbol{\phi}_j \boldsymbol{\phi}_j^{\mathrm{T}}\right)\mathbf{M} + \mathbf{M}\left(\sum_{j=1}^{N} \frac{2\zeta_j \frac{d\omega_j}{d\theta}}{m_j} \boldsymbol{\phi}_j \boldsymbol{\phi}_j^{\mathrm{T}}\right)\mathbf{M} \\
- \mathbf{M}\left(\sum_{j=1}^{N} \frac{2\zeta_j \omega_j}{m_j^2} \cdot \frac{dm_j}{d\theta} \cdot \boldsymbol{\phi}_j \boldsymbol{\phi}_j^{\mathrm{T}}\right)\mathbf{M} + \mathbf{M}\left(\sum_{j=1}^{N} \frac{2\zeta_j \omega_j}{m_j} \cdot \frac{d\boldsymbol{\phi}_j}{d\theta} \boldsymbol{\phi}_j^{\mathrm{T}}\right)\mathbf{M} \\
+ \mathbf{M}\left(\sum_{j=1}^{N} \frac{2\zeta_j \omega_j}{m_j} \boldsymbol{\phi}_j \frac{d\boldsymbol{\phi}_j^{\mathrm{T}}}{d\theta}\right)\mathbf{M} + \mathbf{M}\left(\sum_{j=1}^{N} \frac{2\zeta_j \omega_j}{m_j} \boldsymbol{\phi}_j \boldsymbol{\phi}_j^{\mathrm{T}}\right)\frac{d\mathbf{M}}{d\theta}
\end{aligned} \tag{2}$$

where $dm_j/d\theta = (d\boldsymbol{\phi}_j/d\theta)^{\mathrm{T}}\mathbf{M}\boldsymbol{\phi}_j + \boldsymbol{\phi}_j^{\mathrm{T}}(d\mathbf{M}/d\theta)\boldsymbol{\phi}_j + \boldsymbol{\phi}_j^{\mathrm{T}}\mathbf{M}(d\boldsymbol{\phi}_j/d\theta)$. The derivative of the mass matrix is usually trivial and the derivative of eigenvalues and eigenvectors are addressed in another document posted near this one. Note that once the derivative of an eigenvalue is known the derivative of the natural frequency is

$$\frac{d\omega_j}{d\theta} = \frac{1}{2} \cdot \frac{1}{\sqrt{\gamma_j}} \cdot \frac{d\gamma_j}{d\theta} \tag{3}$$

where γ_j is an eigenvalue of the problem $[\mathbf{K} - \gamma\mathbf{M}]\boldsymbol{\phi}=\mathbf{0}$.

Effect of using the Tangent Stiffness

In nonlinear dynamics, there is a possibility of letting the current tangent stiffness matrix be used when the eigenvalue problem is solved to calculate **C**. In that case, the eigenvalues depend implicitly on θ via **K**, which in turn is a function of **u**. As a result, differentiation of the damping matrix after convergence at increment n+1 yields

$$\frac{\partial \mathbf{C}}{\partial \theta} = \frac{\partial \mathbf{C}}{\partial \mathbf{K}_{n+1}} \cdot \frac{\partial \mathbf{K}_{n+1}}{\partial \mathbf{u}_{n+1}} \cdot \frac{\partial \mathbf{u}_{n+1}}{\partial \theta} + \left. \frac{\partial \mathbf{C}(\mathbf{u}_{n+1})}{\partial \theta} \right|_{\mathbf{u}\text{ fixed}} \tag{4}$$

The last term is essentially presented in Eq. (2). The first term after the equal sign contains the third-order tensor formed by the derivative of the stiffness matrix with respect to the displacement vector. That tensor also appears in the document on derivatives of Rayleigh damping. The first term after the equal sign also contains the fourth-order tensor formed by the derivative of the damping matrix with respect to the stiffness matrix. To continue the derivations, it is helpful to switch to index notation. The contribution to the damping matrix from *one* mode is:

$$C_{ik} = M_{il} \frac{2\zeta\omega}{m} \phi_l \phi_n M_{nk} \tag{5}$$

The derivative of the damping matrix with respect to the stiffness matrix is

$$\frac{\partial C_{ik}}{\partial K_{op}} = \frac{\partial C_{ik}}{\partial m} \cdot \frac{\partial m}{\partial \phi_q} \cdot \frac{\partial \phi_q}{\partial K_{op}} + \frac{\partial C_{ik}}{\partial \omega} \cdot \frac{\partial \omega}{\partial \gamma} \cdot \frac{\partial \gamma}{\partial K_{op}} + \frac{\partial C_{ik}}{\partial \phi_q} \cdot \frac{\partial \phi_q}{\partial K_{op}} \tag{6}$$

where ω is the square root of the eigenvalue γ, so that $d\omega/d\gamma = 1/(2\omega)$. The derivative of eigenvalues and eigenvectors with respect to the stiffness matrix is addressed in the document on eigenvalue derivatives posted on this webpage. Turning to the other derivatives, the derivative of the damping matrix with respect to the modal mass is

$$\frac{\partial C_{ik}}{\partial m} = -M_{il} \frac{2\zeta\omega}{m^2} \phi_l \phi_n M_{nk} \tag{7}$$

Next, the derivative of the modal mass, which reads $\phi_s M_{st} \phi_t$, with respect to each eigenvector is

$$\frac{\partial m}{\partial \phi_q} = M_{qt}\phi_t + \phi_s M_{sq} \tag{8}$$

Now addressing the second term in Eq. (6), the derivative of the damping matrix with respect to the natural frequency is

$$\frac{\partial C_{ik}}{\partial \omega} = M_{il} \frac{2\zeta}{m} \phi_l \phi_n M_{nk} \tag{9}$$

The derivative of the damping matrix with respect to the eigenvector is also obtained by differentiating Eq. (5):

$$\frac{\partial C_{ik}}{\partial \phi_q} = M_{iq}\frac{2\zeta\omega}{m}\phi_n M_{nk} + M_{il}\frac{2\zeta\omega}{m}\phi_l M_{qk} \tag{10}$$

In the right-hand side of the linear system of equations for $d\mathbf{u}/d\theta$ for dynamic problems, the derivative of the damping matrix, i.e., $d\mathbf{C}/d\theta$ Eq. (4) multiplies the parenthesis $(a_5\mathbf{u}_{n+1} + a_6\mathbf{u}_n + a_7\dot{\mathbf{u}}_n + a_8\ddot{\mathbf{u}}_n)$. In index notation, focusing on the first term after the equal sign in Eq. (4), the relevant part of that right-hand side reads

$$\begin{aligned}\frac{\partial \mathbf{C}}{\partial \mathbf{K}_{n+1}} \cdot \frac{\partial \mathbf{K}_{n+1}}{\partial \mathbf{u}_{n+1}} \cdot \frac{\partial \mathbf{u}_{n+1}}{\partial \theta} \cdot (a_5\mathbf{u}_{n+1} + a_6\mathbf{u}_n + a_7\dot{\mathbf{u}}_n + a_8\ddot{\mathbf{u}}_n)\\ = \frac{\partial C_{ik}}{\partial K_{op}} \cdot \frac{\partial K_{op}}{\partial u_r} \cdot \frac{\partial u_r}{\partial \theta} \cdot \mathrm{a5parenthesis}_k\end{aligned} \tag{11}$$

In order to solve the system of equations for $d\mathbf{u}/d\theta$, that quantity must be pulled out from the expression in Eq. (11). The resulting amendment to the coefficient matrix in the linear system of equations for $d\mathbf{u}/d\theta$ is the following expression summed over all modes:

$$\mathrm{amendment}_{ir} = \frac{\partial C_{ik}}{\partial K_{op}} \cdot \frac{\partial K_{op}}{\partial u_r} \cdot \mathrm{a5parenthesis}_k \tag{12}$$

where the parenthesis $(a_5\mathbf{u}_{n+1} + a_6\mathbf{u}_n + a_7\dot{\mathbf{u}}_n + a_8\ddot{\mathbf{u}}_n)$ is contracted with the second index of the fourth-order tensor $\partial\mathbf{C}/\partial\mathbf{K}$. Combining expressions, the amendment reads

$$\begin{aligned}\mathrm{amend}_{ir} &= \frac{\partial C_{ik}}{\partial K_{op}} \cdot \frac{\partial K_{op}}{\partial u_r} \cdot \mathrm{a5}_k\\
&= \left(\frac{\partial C_{ik}}{\partial m} \cdot \frac{\partial m}{\partial \phi_q} \cdot \frac{\partial \phi_q}{\partial K_{op}} + \frac{\partial C_{ik}}{\partial \omega} \cdot \frac{\partial \omega}{\partial \gamma} \cdot \frac{\partial \gamma}{\partial K_{op}} + \frac{\partial C_{ik}}{\partial \phi_q} \cdot \frac{\partial \phi_q}{\partial K_{op}}\right) \cdot \frac{\partial K_{op}}{\partial u_r} \cdot \mathrm{a5}_k\\
&= \left(-M_{il}\frac{2\zeta\omega}{m^2}\phi_l\phi_n M_{nk} \cdot (M_{qt}\phi_t + \phi_s M_{sq}) \cdot \frac{\partial \phi_q}{\partial K_{op}} + M_{il}\frac{2\zeta}{m}\phi_l\phi_n M_{nk} \cdot \frac{1}{2\omega} \cdot \frac{\partial \gamma}{\partial K_{op}} + \left(M_{iq}\frac{2\zeta\omega}{m}\phi_n M_{nk} + M_{il}\frac{2\zeta\omega}{m}\phi_l M_{qk}\right)\frac{\partial \phi_q}{\partial K_{op}}\right) \cdot \frac{\partial K_{op}}{\partial u_r} \cdot \mathrm{a5}_k\\
&= \left(-\frac{2\zeta\omega}{m^2}M_{il}\phi_l\phi_n M_{nk} \cdot (M_{qt}\phi_t + \phi_s M_{sq}) \cdot \frac{\partial \phi_q}{\partial K_{op}} + \frac{2\zeta}{2m\omega}M_{il}\phi_l\phi_n M_{nk} \cdot \frac{\partial \gamma}{\partial K_{op}} + \frac{2\zeta\omega}{m}(M_{iq}\phi_n M_{nk} + M_{il}\phi_l M_{qk})\frac{\partial \phi_q}{\partial K_{op}}\right) \cdot \frac{\partial K_{op}}{\partial u_r} \cdot \mathrm{a5}_k\\
&= \left(-\frac{2\zeta\omega}{m^2}b_i b_k \cdot (b_q + b_q) \cdot \frac{\partial \phi_q}{\partial K_{op}} + \frac{2\zeta}{2m\omega}b_i b_k\frac{\partial \gamma}{\partial K_{op}} + \frac{2\zeta\omega}{m}(M_{iq}b_k + b_i M_{qk})\frac{\partial \phi_q}{\partial K_{op}}\right) \cdot \frac{\partial K_{op}}{\partial u_r} \cdot \mathrm{a5}_k\\
&= \left(-\frac{4\zeta\omega}{m^2}b_i b_k b_q\frac{\partial \phi_q}{\partial K_{op}} + \frac{\zeta}{m\omega}b_i b_k\frac{\partial \gamma}{\partial K_{op}} + \frac{2\zeta\omega}{m}(M_{iq}b_k + b_i M_{qk})\frac{\partial \phi_q}{\partial K_{op}}\right) \cdot \frac{\partial K_{op}}{\partial u_r} \cdot \mathrm{a5}_k\\
&= \left(-\frac{4\zeta\omega}{m^2}b_i b_k b_q\frac{\partial \phi_q}{\partial K_{op}} + \frac{\zeta}{m\omega}b_i b_k\frac{\partial \gamma}{\partial K_{op}} + \frac{2\zeta\omega}{m}(M_{iq}b_k + b_i M_{qk})\frac{\partial \phi_q}{\partial K_{op}}\right) \cdot \frac{\partial K_{op}}{\partial u_r} \cdot \mathrm{a5}_k\end{aligned} \tag{13}$$

where $b_i = M_{ij}\phi_j$ and summations for i, k, and q is only needed for the degrees of freedom that have mass.

References

Chopra, A. K. & McKenna, F. (2016). "Modeling viscous damping in nonlinear response history analysis of buildings for earthquake excitation." Earthquake Engineering & Structural Dynamics, 45, pp. 193-211.