DDM for Linear Static Problems

The direct differentiation method, DDM, involves the differentiation of the equations and algorithms that govern the response. For linear static problems, the governing equilibrium equations are

$$\mathbf{K}\mathbf{u} = \mathbf{F} \tag{1}$$

where **K**=stiffness matrix, **u**=vector of displacements and rotations, and **F**=load vector. The symbol θ represents some parameter that the response is dependent upon. Examples are material parameters in **K** and loads in **F**. That is recognized by writing Eq. (1) as

$$\mathbf{K}(\theta)\mathbf{u}(\theta) = \mathbf{F}(\theta) \tag{2}$$

The objective is to determine the response sensitivity $\partial \mathbf{u}/\partial \theta$. Differentiating Eq. (2) with respect to θ yields, by the product rule of differentiation:

$$\frac{\partial \mathbf{K}}{\partial \theta}\mathbf{u} + \mathbf{K}\frac{\partial \mathbf{u}}{\partial \theta} = \frac{\partial \mathbf{F}}{\partial \theta} \tag{3}$$

Rearranging yields

$$\mathbf{K}\frac{\partial \mathbf{u}}{\partial \theta} = \frac{\partial \mathbf{u}}{\partial \theta} - \frac{\partial \mathbf{u}}{\partial \theta}\mathbf{u} \tag{4}$$

which is a linear system of the same form as **Ku**=**F**, albeit with a different right-hand side. Eq. (4) is implemented in the finite element code alongside Eq. (1) in order to obtain response sensitivities. Once $\partial \mathbf{u}/\partial \theta$ is obtained it is straightforward to obtain the derivative of responses that are a function of **u**, such as stresses and strains. Notice that, for response sensitivity analysis, it is necessary to assemble $\partial \mathbf{K}/\partial \theta$ alongside the assembly of **K** and it is necessary to assemble $\partial \mathbf{F}/\partial \theta$ alongside the assembly of **F**. In situations where $\partial u/\partial \theta$ is required for only one or a few components of **u**, the adjoint method, described in another document, circumvents the need for calculating the full vector $\partial \mathbf{u}/\partial \theta$.