DDM for Linear Dynamic Problems

The equations presented in this document are implemented in the programming language Python in the G2 code posted on this website. The governing system of equations for this class of problems is

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{C}\dot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \mathbf{F} \tag{1}$$

where **M**=mass matrix, **u**=displacement vector, **C**=damping matrix, **K**=stiffness matrix, **F**=applied loads. Each dot above a symbol means one derivative with respect to time. In the following, the symbol θ represents a parameter that the response is dependent upon. Examples are material properties, such as stiffness and density variables, and applied loads. The primary objective is to obtain $\partial\mathbf{u}/\partial\theta$. The dependence on θ is shown by writing Eq. (1) as

$$\mathbf{M}(\theta)\ddot{\mathbf{u}}(\theta) + \mathbf{C}(\theta)\dot{\mathbf{u}}(\theta) + \mathbf{K}(\theta)\mathbf{u}(\theta) = \mathbf{F}(\theta) \tag{2}$$

The equilibrium equations in Eq. (2) are already discretized in space. That is done by means of degrees of freedom connected by finite elements, whose contributions to **M** and **K** are obtained by integration. To derive equations for $\partial\mathbf{u}/\partial\theta$, the equilibrium equations are now discretized in time. A generic time-stepping scheme is

$$\begin{aligned} \ddot{\mathbf{u}}_{n+1} &= a_1 \cdot \mathbf{u}_{n+1} + a_2 \cdot \mathbf{u}_n + a_3 \cdot \dot{\mathbf{u}}_n + a_4 \cdot \ddot{\mathbf{u}}_n \\ \dot{\mathbf{u}}_{n+1} &= a_5 \cdot \mathbf{u}_{n+1} + a_6 \cdot \mathbf{u}_n + a_7 \cdot \dot{\mathbf{u}}_n + a_8 \cdot \ddot{\mathbf{u}}_n \end{aligned} \tag{3}$$

where n counts over the time steps and the a-constants are selected to match some time-stepping scheme. The Newmark algorithm with β=0.25 and γ=0.5 is written

- $a_1 = 1/(\beta\,\Delta t^2)$
- $a_2 = -a_1$
- $a_3 = -1.0/(\beta\,\Delta t)$
- $a_4 = 1.0-1/(2\beta)$
- $a_5 = \gamma/(\beta\,\Delta t)$
- $a_6 = -a_5$
- $a_7 = 1-\gamma/\beta$
- $a_8 = \Delta t\,(1-\gamma/(2\beta))$

Eq. (2) written at time step n+1 reads

$$\mathbf{M}\ddot{\mathbf{u}}_{n+1} + \mathbf{C}\dot{\mathbf{u}}_{n+1} + \mathbf{K}\mathbf{u}_{n+1} = \mathbf{F}_{n+1} \tag{4}$$

Substitution of Eq. (3) into Eq. (4) yields

$$\begin{aligned} \mathbf{M}(a_1\mathbf{u}_{n+1} + a_2\mathbf{u}_n + a_3\dot{\mathbf{u}}_n + a_4\ddot{\mathbf{u}}_n) + \mathbf{C}(a_5\mathbf{u}_{n+1} + a_6\mathbf{u}_n + a_7\dot{\mathbf{u}}_n \\ + a_8\ddot{\mathbf{u}}_n) + \mathbf{K}\mathbf{u}_{n+1} = \mathbf{F}_{n+1} \end{aligned} \tag{5}$$

Eq. (5) represents the spatially and temporally discretized equilibrium equations that govern the response at time n+1, i.e., $\mathbf{u}_{n+1}$. It is a linear system of equations, solved for $\mathbf{u}_{n+1}$ by rearranging Eq. (5) as follows:

$$\begin{gathered}(a_1\mathbf{M}+a_5\mathbf{C}+\mathbf{K})\mathbf{u}_{n+1}\\ =\mathbf{F}_{n+1}-\mathbf{M}(a_2\mathbf{u}_n+a_3\dot{\mathbf{u}}_n+a_4\ddot{\mathbf{u}}_n)-\mathbf{C}(a_6\mathbf{u}_n+a_7\dot{\mathbf{u}}_n+a_8\ddot{\mathbf{u}}_n)\end{gathered} \tag{6}$$

where $a_1\mathbf{M}+a_5\mathbf{C}+\mathbf{K}$ in the left-hand side is referred to as the effective tangent stiffness. To calculate $\partial\mathbf{u}_{n+1}/\partial\theta$ by the direct differentiation method, Eq. (5) is differentiated with respect to θ:

$$\begin{gathered}\frac{\partial\mathbf{M}}{\partial\theta}(a_1\mathbf{u}_{n+1}+a_2\mathbf{u}_n+a_3\dot{\mathbf{u}}_n+a_4\ddot{\mathbf{u}}_n)\\ +\mathbf{M}\left(a_1\frac{\partial\mathbf{u}_{n+1}}{\partial\theta}+a_2\frac{\partial\mathbf{u}_n}{\partial\theta}+a_3\frac{\partial\dot{\mathbf{u}}_n}{\partial\theta}+a_4\frac{\partial\ddot{\mathbf{u}}_n}{\partial\theta}\right)+\frac{\partial\mathbf{C}}{\partial\theta}(a_5\mathbf{u}_{n+1}+a_6\mathbf{u}_n\\ +a_7\dot{\mathbf{u}}_n+a_8\ddot{\mathbf{u}}_n)+\mathbf{C}(a_5\frac{\partial\mathbf{u}_{n+1}}{\partial\theta}+a_6\frac{\partial\mathbf{u}_n}{\partial\theta}+a_7\frac{\partial\dot{\mathbf{u}}_n}{\partial\theta}+a_8\frac{\partial\ddot{\mathbf{u}}_n}{\partial\theta})\\ +\frac{\partial\mathbf{K}}{\partial\theta}\mathbf{u}_{n+1}+\mathbf{K}\frac{\partial\mathbf{u}_{n+1}}{\partial\theta}=\frac{\partial\mathbf{F}_{n+1}}{\partial\theta}\end{gathered} \tag{7}$$

The derivative of the damping matrix, $\partial\mathbf{C}/\partial\theta$, is addressed in other documents on the webpage where this one is posted. When the loading is ground acceleration, the load derivative is

$$\frac{\partial\mathbf{F}_{n+1}}{\partial\theta}=-\frac{d\mathbf{M}}{d\theta}\cdot\mathbf{\Gamma}\cdot\begin{Bmatrix}\ddot{u}_{gx}\\ \ddot{u}_{gy}\\ \ddot{u}_{gz}\end{Bmatrix} \tag{8}$$

Rearranging Eq. (7) by collecting terms that do, or do not, multiply the unknown vector, i.e., $\frac{\partial\mathbf{u}_{n+1}}{\partial\theta}$, yields

$$\begin{gathered}(a_1\mathbf{M}+a_5\mathbf{C}+\mathbf{K})\frac{\partial\mathbf{u}_{n+1}}{\partial\theta}\\ =\frac{\partial\mathbf{F}_{n+1}}{\partial\theta}-\frac{\partial\mathbf{M}}{\partial\theta}(a_1\mathbf{u}_{n+1}+a_2\mathbf{u}_n+a_3\dot{\mathbf{u}}_n+a_4\ddot{\mathbf{u}}_n)\\ -\mathbf{M}\left(a_2\frac{\partial\mathbf{u}_n}{\partial\theta}+a_3\frac{\partial\dot{\mathbf{u}}_n}{\partial\theta}+a_4\frac{\partial\ddot{\mathbf{u}}_n}{\partial\theta}\right)\\ -\frac{\partial\mathbf{C}}{\partial\theta}(a_5\mathbf{u}_{n+1}+a_6\mathbf{u}_n+a_7\dot{\mathbf{u}}_n+a_8\ddot{\mathbf{u}}_n)\\ -\mathbf{C}\left(a_6\frac{\partial\mathbf{u}_n}{\partial\theta}+a_7\frac{\partial\dot{\mathbf{u}}_n}{\partial\theta}+a_8\frac{\partial\ddot{\mathbf{u}}_n}{\partial\theta}\right)-\frac{\partial\mathbf{K}}{\partial\theta}\mathbf{u}_{n+1}\end{gathered} \tag{9}$$

Notice that the coefficient matrix of the linear system of equations in Eq. (9) is identical to that of the response equations in Eq. (6). The derivatives of **M**, **C**, and **K** are assembled from element contributions in a manner similar to the assembly of **M**, **C**, and **K**, according to equations in documents posted near this one.