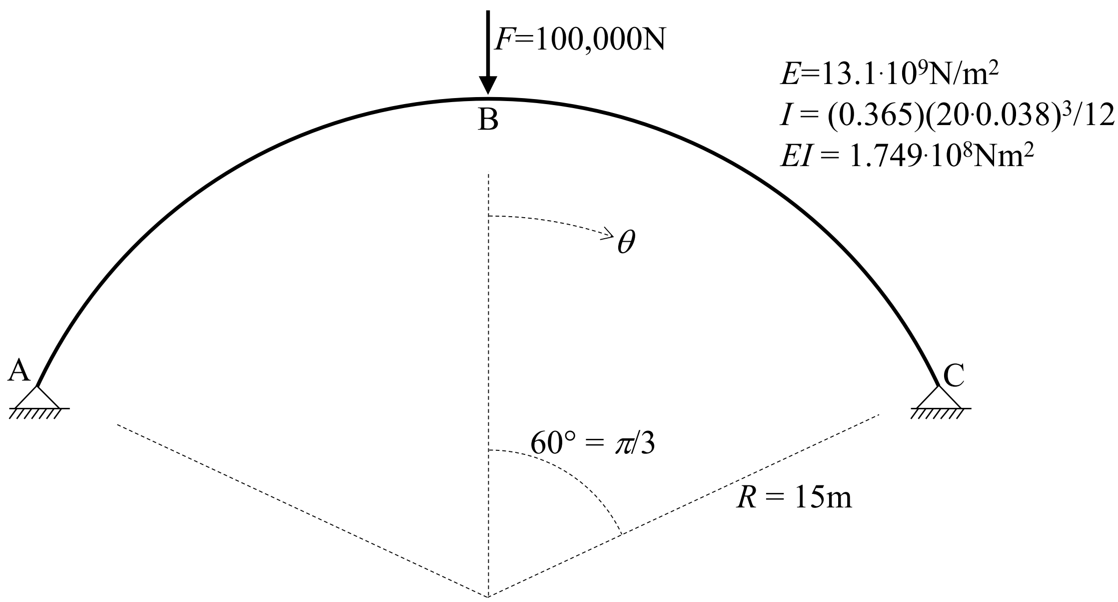
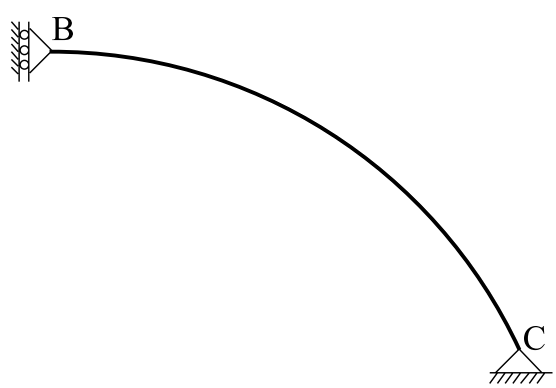
Indeterminate Curved Beam

Consider the statically indeterminate arc below. It is made of a 365mm wide glulam beam with 20 laminates, each with 38mm thickness. Its load and radius are also given in the figure. The task is to determine the bending moment diagram (BMD) using the flexibility method with the bending moment at the centre of the arch as the redundant. Axial deformations are to be neglected.



The moment at B is the selected redundant. Therefore, we introduce a hinge at B in order to make the structure statically determinate. Furthermore, we take advantage of symmetry, so that the structure we analyze is this one:

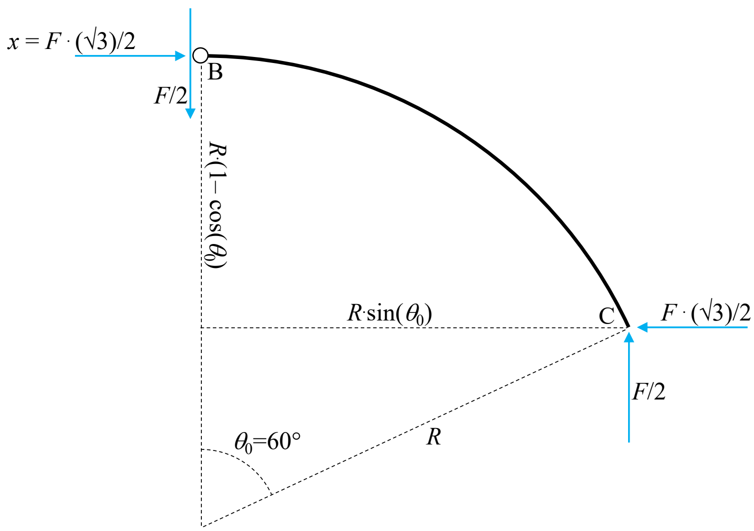


In the subsequent calculations, notice that $\cos(\pi/3)=1/2$ and $\sin(\pi/3)=\sqrt{3}/2$.
We start by analyzing that structure for the external load, which will give the M_0 diagram. Half the total load, F, is taken by member AB. The other half is taken by member BC.

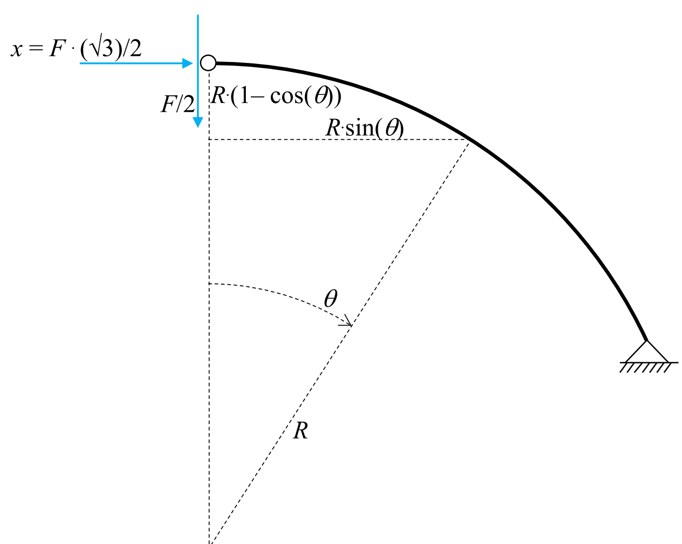
That means the vertical force at B is $F/2$=50kN. By sum of vertical forces we obtain the vertical reaction force at C to be the same value. Sum of moments about C gives the horizontal reaction force at B:

$$\Sigma M_C = x \cdot R \cdot (1 - \cos(\pi/3)) - F/2 \cdot R \cdot \sin(\pi/3) = 0 \rightarrow x = (F \cdot R \cdot \sin(\pi/3)) / (2 \cdot R \cdot (1 - \cos(\pi/3))$$

Thereafter, the sum of horizontal forces gives the horizontal reaction at C. The result is:

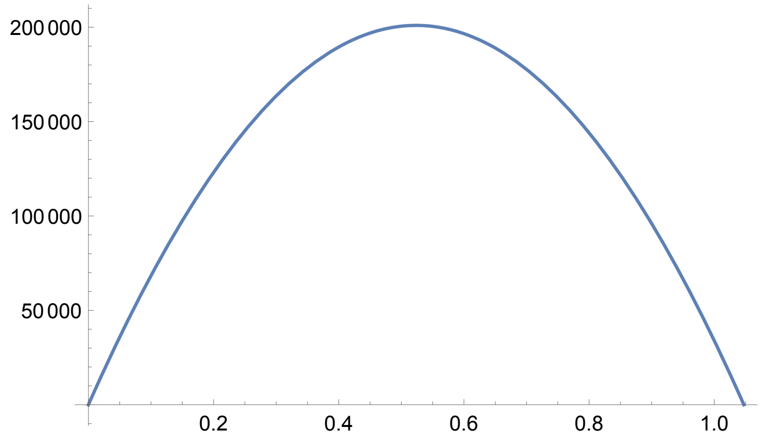


By considering a "cut" at a location identified by the angle θ, we obtain the sought bending moment diagram, M_0:



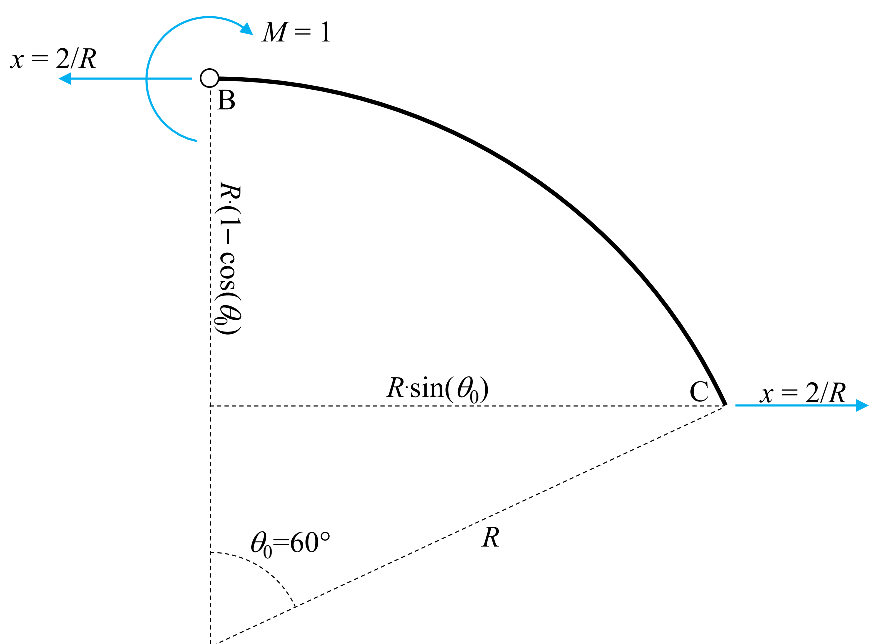
$$M_0(\theta) = F \cdot (\sqrt{3})/2 \cdot R \cdot (1 - \cos(\theta)) - F/2 \cdot R \cdot \sin(\theta)$$

Here is a plot of M_0 from θ from 0 to $\pi/3$:



Next, we analyze the determinate structure for a unit moment along the redundant, in order to obtain the bending moment diagram M_B. Notice that the shear force at B is zero by symmetry. Sum of vertical forces means that the vertical reaction at C is also zero. Sum of moments about C gives the horizontal force at B:

$$\Sigma M_C = x \cdot R \cdot (1 - \cos(\theta_0)) - M = 0 \rightarrow x = 1 / (R \cdot (1 - \cos(\theta_0)) = 1 / (R \cdot (1 - ½) = 2/R$$



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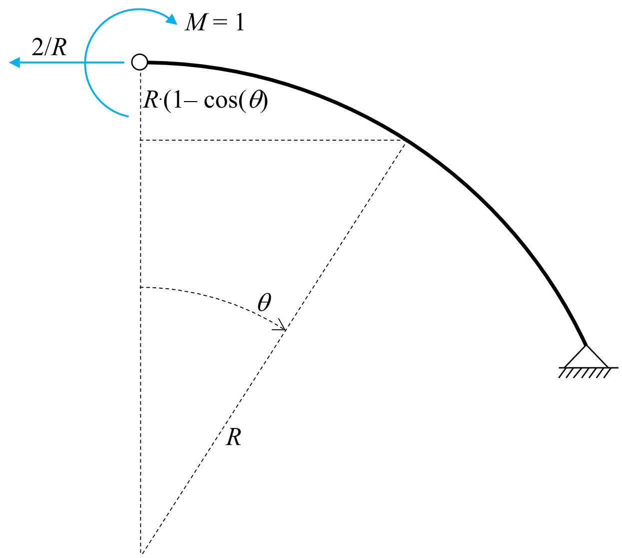
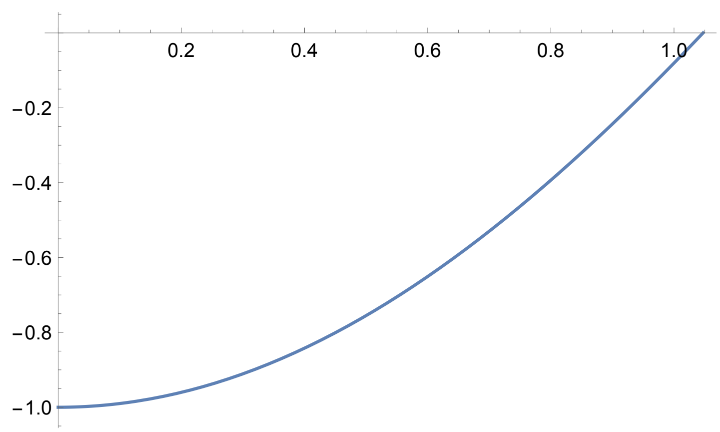


$$M_B(\theta) = 1 - 2/R \cdot R \cdot (1 - \cos(\theta)) = 1 - 2\,(1 - \cos(\theta))$$

Here is a plot of M_B from θ from 0 to $\pi/3$:



Having M0 and MB, we are ready to use virtual work to calculate the rotations that appear in the compatibility equation $\theta_{B0} + \theta_{BB} X_B = 0$:

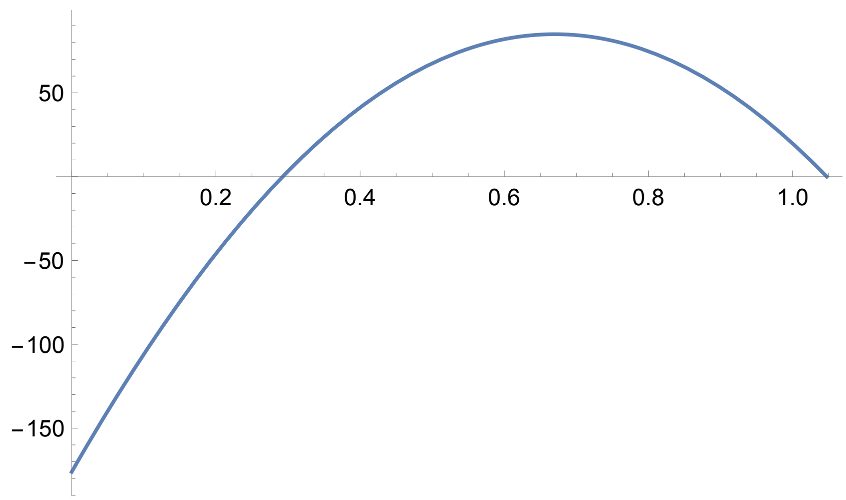
$$\theta_{B0} = \int_0^{\frac{\pi}{3}} M_0 \cdot \frac{M_B}{EI}\, d\theta = -0.000547122$$

$$\theta_{BB} = \int_0^{\frac{\pi}{3}} M_B \cdot \frac{M_B}{EI}\, d\theta = 3.10734 \cdot 10^{-9}$$

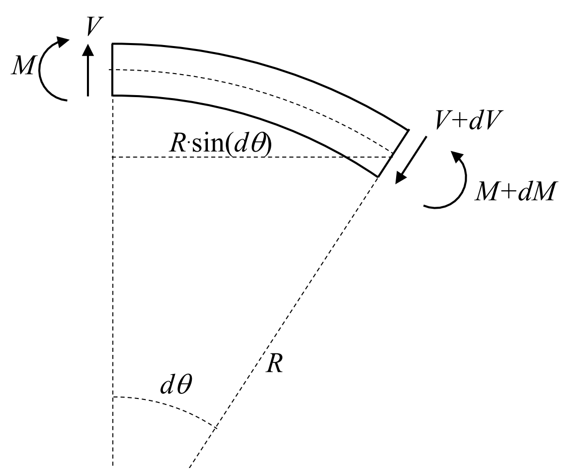
Solving the compatibility equation yields:

$$X_{\mathrm{B}} = -\frac{\theta_{\mathrm{B0}}}{\theta_{\mathrm{BB}}} = 176.07\mathrm{kNm}$$

That is the final value of the bending moment at B. The rest of the bending moment diagram looks like this, plotted for θ from 0 to $\pi/3$:



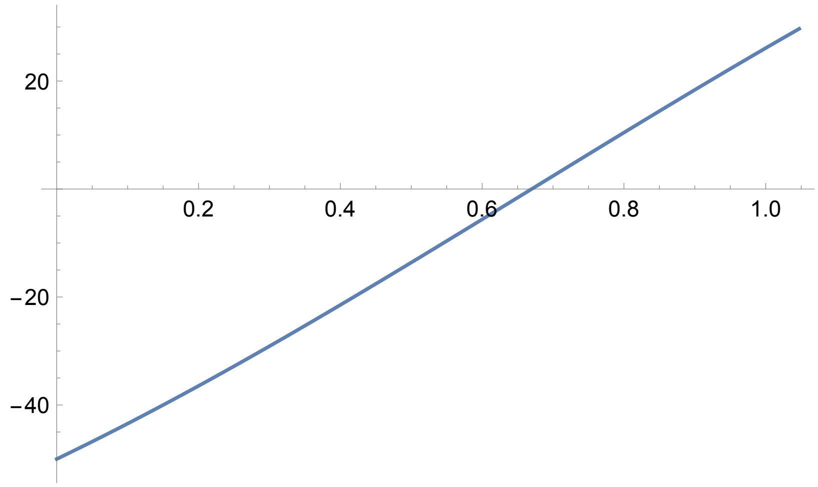
Now a comment about the shear force. At first glance, we may think it is $V=dM/d\theta$, as it nominally is for straight beams. However, this is the proper equilibrium consideration in order to determine the shear force in the curved beam:



Taking the sum of moments about the right-hand side edge gives, because $\sin(d\theta)\approx d\theta$:

$$\sum M = M - M - \mathrm{d}M + V \cdot R \cdot \sin(d\theta) = 0 \rightarrow V = (1/R) \cdot dM/d\theta$$

As a result, here is a plot of V, in kN, from θ from 0 to $\pi/3$:


20
0.2
0.4
0.6
0.8
1.0
−20
−40