# Eigenvalue Derivatives

Eigenvalue derivatives are needed when then direct differentiation method is applied to dynamic analysis with modal damping. The derivatives developed in this document are also needed in Rayleigh damping if the coefficients $c_M$ and $c_K$ are determined from a target damping ratio specified at two of the natural frequencies of the structure. That is because the natural frequencies will change if the mass or stiffness change, in turn influencing $c_M$ and $c_K$. Consider the generalized eigenvalue problem of multi-degree-of-freedom structural dynamics:

$$[\mathbf{K}-\gamma\mathbf{M}]\boldsymbol{\phi}=\mathbf{0} \tag{1}$$

where $\mathbf{K}$ is the stiffness matrix, $\gamma$ is an eigenvalue, $\mathbf{M}$ is the mass matrix, and $\boldsymbol{\phi}$ is an eigenvector. The corresponding natural frequency of vibration is the square root of $\gamma$. The number of solutions, i.e., the number of eigenvalues and eigenvectors, equals the number of degrees of freedom, i.e., the dimension of $\mathbf{K}$ and $\mathbf{M}$.

### Eigenvalue Derivatives

In order to obtain the derivative of $\gamma$ it turns out to be necessary to be conscious of how the corresponding eigenvector, $\boldsymbol{\phi}$, is scaled. For example, if the eigenvector is scaled to unit length, which is done here, then the norm of $\boldsymbol{\phi}$ does not change with $\theta$. This is made use of by first recognizing that the norm reads

$$\|\boldsymbol{\phi}\|=(\phi_1^2+\phi_2^2+\cdots)^{\frac{1}{2}} \tag{2}$$

That means the derivative of the norm, which is zero as mentioned above, reads

$$\frac{d\|\boldsymbol{\phi}\|}{d\theta}=\frac{1}{2}(\phi_1^2+\phi_2^2+\cdots)^{-\frac{1}{2}}\cdot\frac{d(\phi_1^2+\phi_2^2+\cdots)}{d\theta} \tag{3}$$

The last factor evaluates to

$$\frac{d(\phi_1^2+\phi_2^2+\cdots)}{d\theta}=2\cdot\phi_1\cdot\frac{d\phi_1}{d\theta}+2\cdot\phi_2\cdot\frac{d\phi_2}{d\theta}+\cdots \tag{4}$$

That means the derivative of the norm is

$$\frac{d\|\boldsymbol{\phi}\|}{d\theta}=\frac{1}{2}\cdot\frac{1}{\|\boldsymbol{\phi}\|}\cdot\left(2\,\phi_1\frac{d\phi_1}{d\theta}+2\,\phi_2\frac{d\phi_2}{d\theta}+\cdots\right) \tag{5}$$

Because the factor 2 in the denominator cancels with the 2 in the numerator, and because $\|\boldsymbol{\phi}_n\|=1$, the derivative of the norm is the dot product

$$\frac{d\|\boldsymbol{\phi}\|}{d\theta}=\boldsymbol{\phi}^{\mathrm{T}}\frac{d\boldsymbol{\phi}}{d\theta} \tag{6}$$

The fact that Eq. (5) evaluates to zero, as mentioned earlier, will be employed shortly. In order to obtain $\partial\gamma/\partial\theta$ and $\partial\boldsymbol{\phi}/\partial\theta$, where $\theta$ is some parameter in **K** or **M** or both, Eq. (1) is differentiated with respect to $\theta$. The product rule of differentiation leads to

$$\frac{d\mathbf{K}}{d\theta}\boldsymbol{\phi} - \frac{d\gamma}{d\theta}\mathbf{M}\,\boldsymbol{\phi} - \gamma\frac{d\mathbf{M}}{d\theta}\boldsymbol{\phi} + [\mathbf{K} - \gamma\,\mathbf{M}]\,\frac{d\boldsymbol{\phi}}{d\theta} = \mathbf{0} \tag{7}$$

Eq. (7) is now multiplied through by the transposed of $\boldsymbol{\phi}$ from the left:

$$\boldsymbol{\phi}^{\mathrm{T}}\frac{d\mathbf{K}}{d\theta}\boldsymbol{\phi} - \boldsymbol{\phi}^{\mathrm{T}}\frac{d\gamma}{d\theta}\mathbf{M}\,\boldsymbol{\phi} - \boldsymbol{\phi}^{\mathrm{T}}\gamma\frac{d\mathbf{M}}{d\theta}\boldsymbol{\phi} + \boldsymbol{\phi}^{\mathrm{T}}[\mathbf{K} - \gamma\,\mathbf{M}]\frac{d\boldsymbol{\phi}}{d\theta} = \mathbf{0} \tag{8}$$

Because of symmetry of **M** and **K**, Eq. (5) causes the last term to vanish, leading to the following equation for $\partial\gamma/\partial\theta$:

$$\frac{d\gamma}{d\theta} = \frac{\boldsymbol{\phi}^{\mathrm{T}}\frac{d\mathbf{K}}{d\theta}\boldsymbol{\phi} - \gamma\boldsymbol{\phi}^{\mathrm{T}}\frac{d\mathbf{M}}{d\theta}\boldsymbol{\phi}}{\boldsymbol{\phi}^{\mathrm{T}}\mathbf{M}\,\boldsymbol{\phi}} \tag{9}$$

## Eigenvector Derivatives

The derivative of the eigenvector is addressed by returning to Eq. (7). Solving for the term with the sought derivative yields

$$[\mathbf{K} - \gamma\,\mathbf{M}]\,\frac{d\boldsymbol{\phi}}{d\theta} = \left[\frac{d\gamma}{d\theta}\mathbf{M} + \gamma\frac{d\mathbf{M}}{d\theta} - \frac{d\mathbf{K}}{d\theta}\right]\boldsymbol{\phi} \tag{10}$$

Multiplying through from the left side by the pseudoinverse of $[\mathbf{K}-\gamma\mathbf{M}]$ yields (Magnus 1985)

$$\frac{d\boldsymbol{\phi}}{d\theta} = [\mathbf{K} - \gamma\,\mathbf{M}]^{+}\left[\frac{d\gamma}{d\theta}\mathbf{M} + \gamma\frac{d\mathbf{M}}{d\theta} - \frac{d\mathbf{K}}{d\theta}\right]\boldsymbol{\phi} \tag{11}$$

The point that $\boldsymbol{\phi}$ is scaled to unit length is reiterated here for two reasons. First, that scaling should be performed before evaluating Eq. (11). Second, in order to compare the result of Eq. (11) with finite difference calculations, the eigenvectors must always be scaled. Otherwise, for programs that return eigenvectors with one component set to unity, that component would always show zero sensitivity in finite difference calculations.

## Derivatives with respect to the Stiffness Matrix

Calculation of derivatives with respect to the stiffness matrix is motivated by the need for $\partial\gamma/\partial\mathbf{K}$ when the Rayleigh damping coefficients are functions of two natural frequencies. Differentiation of Eq. (1) yields

$$\frac{d\mathbf{K}}{d\mathbf{K}}\boldsymbol{\phi} - \frac{d\gamma}{d\mathbf{K}}\mathbf{M}\,\boldsymbol{\phi} - \gamma\frac{d\mathbf{M}}{d\mathbf{K}}\boldsymbol{\phi} + [\mathbf{K} - \gamma\mathbf{M}]\frac{d\boldsymbol{\phi}}{d\mathbf{K}} = \mathbf{0} \tag{12}$$

Multiplying through by $\boldsymbol{\phi}^{\mathrm{T}}$ makes the last term vanish, as earlier. The remaining terms are

$$\boldsymbol{\phi}^{\mathrm{T}}\frac{d\mathbf{K}}{d\mathbf{K}}\boldsymbol{\phi} - \boldsymbol{\phi}^{\mathrm{T}}\frac{d\gamma}{d\mathbf{K}}\mathbf{M}\,\boldsymbol{\phi} - \boldsymbol{\phi}^{\mathrm{T}}\gamma\frac{d\mathbf{M}}{d\mathbf{K}}\boldsymbol{\phi} = \mathbf{0} \tag{13}$$

where $d\mathbf{K}/d\mathbf{K}$ is the four-dimensional identity tensor, meaning that the first term is the outer product of the eigenvector with itself, and the derivative of mass with respect to stiffness is assumed to be zero. Solving for the sought derivative yields

$$\frac{d\gamma}{d\mathbf{K}} = \frac{\boldsymbol{\phi}\boldsymbol{\phi}^{\mathrm{T}}}{\boldsymbol{\phi}^{\mathrm{T}}\mathbf{M}\,\boldsymbol{\phi}} \tag{14}$$

The derivative of the eigenvector with respect to the stiffness matrix is addressed by revisiting Eq. (12). Similar to what was done to obtain the derivative with respect to $\theta$ in Eq. (11), multiplying through from the left side by the pseudoinverse of $[\mathbf{K}-\gamma\mathbf{M}]$ and neglecting the derivative of mass with respect to stiffness yields

$$\frac{d\boldsymbol{\phi}}{d\mathbf{K}} = [\mathbf{K} - \gamma\mathbf{M}]^{+}\left[\frac{d\mathbf{K}}{d\mathbf{K}} - \frac{d\gamma}{d\mathbf{K}}\mathbf{M}\right]\boldsymbol{\phi} \tag{15}$$

The derivatives presented in this document are implemented in Python in the G2 code for inelastic dynamic analysis posted on this website. The Rayleigh damping model and the modal damping model employ some or all of these derivatives, depending on the specific damping option.

### References

Magnus, J.R. (1985). "On differentiating eigenvalues and eigenvectors." Econometric Theory, Vol. 1, pp. 179-191.