Bouc-Wen Material Derivatives

The algorithm to calculate the stress for a given strain history for the uniaxial Bouc-Wen material model is presented elsewhere on this website. This document addresses response sensitivity analysis with the direct differentiation method and, initially, the objective is twofold. The first objective is to calculate the conditional stress derivative $(\partial\sigma/\partial\theta)|_{\varepsilon \text{ fixed}}$. This is referred to as "Phase 1" of the direct differentiation method. The second objective is to calculate the unconditional derivative of the history variable of the Bouc-Wen model, i.e., $\partial z/\partial\theta$. This is referred to as "Phase 2," which is carried out once the strain derivative, $\partial\varepsilon/\partial\theta$, is available after the completion of Phase 1. Additional objectives related to Rayleigh damping are addressed towards the end of this document. Three of the input parameters to the Bouc-Wen model are considered as options for θ, namely, E, f_y, or α. Importantly, the calculation of $(\partial\sigma/\partial\theta)|_{\varepsilon \text{ fixed}}$ in Phase 1 and the storage of $\partial z/\partial\theta$ and $\partial\varepsilon/\partial\theta$ in Phase 2 must be carried out regardless of whether θ represents any of the parameters of an instance of the Bouc-Wen model. For instance, if θ represents an element load, the sensitivity history variables $\partial z/\partial\theta$ and $\partial\varepsilon/\partial\theta$ will still be non-zero. For that reason, the material model must store a vector for $\partial z/\partial\theta$ and a vector for $\partial\varepsilon/\partial\theta$, of dimension equal to the total number of variables with respect to which response sensitivities are sought. To simplify the subsequent equations, the following if-statements are first executed

$$
\begin{gathered}
dE/d\theta=1,\ df_y/d\theta=0,\ d\alpha/d\theta=0,\ d\varepsilon_y/d\theta=-f_y/E^2 \text{ if } \theta \text{ is } E \\
dE/d\theta=0,\ df_y/d\theta=1,\ d\alpha/d\theta=0,\ d\varepsilon_y/d\theta=1/E \text{ if } \theta \text{ is } f_y \\
dE/d\theta=0,\ df_y/d\theta=0,\ d\alpha/d\theta=1,\ d\varepsilon_y/d\theta=0 \text{ if } \theta \text{ is } \alpha
\end{gathered}
\tag{1}
$$

The current stress, at increment n+1, is calculated by

$$\sigma_{n+1} = \alpha \cdot E \cdot \varepsilon_{n+1} + (1-\alpha) \cdot f_y \cdot z_{n+1} \tag{2}$$

As a result, the conditional stress derivative is

$$
\begin{gathered}
\left.\frac{d\sigma_{n+1}}{d\theta}\right|_{\varepsilon_{n+1} \text{ fixed}} = \frac{d\alpha}{d\theta} \cdot E \cdot \varepsilon_{n+1} + \alpha \cdot \frac{dE}{d\theta} \cdot \varepsilon_{n+1} \\
-\frac{d\alpha}{d\theta} \cdot f_y \cdot z_{n+1} + (1-\alpha) \cdot \frac{df_y}{d\theta} \cdot z_{n+1} + (1-\alpha) \cdot f_y \cdot \left.\frac{dz_{n+1}}{d\theta}\right|_{\varepsilon_{n+1} \text{ fixed}}
\end{gathered}
\tag{3}
$$

To assist the further derivations, it is recalled that the hysteresis evolution variable in Eq. (2), i.e., z, is obtained by solving the equation $f(z_{n+1})=0$, using the Newton algorithm, where

$$f = z_{n+1} - z_n - \left(1 - \left(\gamma + \beta \cdot \text{sign}(\Delta\varepsilon_{n+1} \cdot z_{n+1})\right) \cdot |z_{n+1}|^\eta\right) \cdot \frac{\varepsilon_{n+1} - \varepsilon_n}{\varepsilon_y} \tag{4}$$

As a result, at the converged state at increment n+1, the hysteresis evolution variable, which is a history variable for this material model, reads

$$z_{n+1} = z_n + \left(1 - \left(\gamma + \beta \cdot \text{sign}(\Delta\varepsilon_{n+1} \cdot z_{n+1})\right) \cdot |z_{n+1}|^\eta\right) \cdot \frac{\varepsilon_{n+1} - \varepsilon_n}{\varepsilon_y} \tag{5}$$

Differentiating Eq. (5) with respect to θ yields

$$\begin{aligned}
\frac{dz_{n+1}}{d\theta} &= \frac{dz_n}{d\theta} - \left(\gamma + \beta \cdot \text{sign}(\Delta\varepsilon_{n+1} \cdot z_{n+1})\right) \cdot \eta \cdot |z_{n+1}|^{\eta-1} \cdot \text{sign}(z_{n+1}) \cdot \frac{dz_{n+1}}{d\theta} \cdot \frac{\varepsilon_{n+1} - \varepsilon_n}{\varepsilon_y} \\
&\quad + \left(1 - \left(\gamma + \beta \cdot \text{sign}(\Delta\varepsilon_{n+1} \cdot z_{n+1})\right) \cdot |z_{n+1}|^\eta\right) \cdot \frac{\frac{d\varepsilon_{n+1}}{d\theta} - \frac{d\varepsilon_n}{d\theta}}{\varepsilon_y} \\
&\quad - \left(1 - \left(\gamma + \beta \cdot \text{sign}(\Delta\varepsilon_{n+1} \cdot z_{n+1})\right) \cdot |z_{n+1}|^\eta\right) \cdot \frac{\varepsilon_{n+1} - \varepsilon_n}{\varepsilon_y^2} \cdot \frac{d\varepsilon_y}{d\theta}
\end{aligned} \tag{6}$$

where $d\varepsilon_{n+1}/d\theta$ is kept in order to facilitate Phase 2 of the direct differentiation method; that strain derivative is set equal to zero in order to obtain the conditional derivative sought in Phase 1. Solving Eq. (6) for $dz_{n+1}/d\theta$ yields

$$\frac{dz_{n+1}}{d\theta} = \frac{\frac{dz_n}{d\theta} - \frac{\frac{d\varepsilon_y}{d\theta}(\varepsilon_{n+1} - \varepsilon_n)\left(1 - \left(\gamma + \beta \cdot \text{sign}(\Delta\varepsilon_{n+1} \cdot z_{n+1})\right) \cdot |z_{n+1}|^\eta\right)}{\varepsilon_y^2} + \frac{\left(\frac{d\varepsilon_{n+1}}{d\theta} - \frac{d\varepsilon_n}{d\theta}\right)\left(1 - \left(\gamma + \beta \cdot \text{sign}(\Delta\varepsilon_{n+1} \cdot z_{n+1})\right) \cdot |z_{n+1}|^\eta\right)}{\varepsilon_y}}{1 + \frac{(\varepsilon_{n+1} - \varepsilon_n)\left(\gamma + \beta \cdot \text{sign}(\Delta\varepsilon_{n+1} \cdot z_{n+1})\right) \cdot \eta \cdot |z_{n+1}|^{\eta-1} \cdot \text{sign}(z_{n+1})}{\varepsilon_y}} \tag{7}$$

Eq. (7) is implemented in G2 on this website, with $d\varepsilon_{n+1}/d\theta = 0$ in Phase 1 of the direct differentiation method and $d\varepsilon_{n+1}/d\theta$ non-zero and passed through the elements and cross-sections to the material model in Phase 2. Notice that Eq. (7) must be evaluated both in Phase 1 and Phase 2, even if the parameter θ is not a parameter of the material model. Otherwise, incorrect response sensitivity results would be produced by the direct differentiation method.

Derivatives of the Tangent Stiffness

When Rayleigh damping with the current tangent stiffness is utilized in nonlinear dynamic analysis then that stiffness appears directly in the equations of motion. In turn, that means the derivative of the current tangent stiffness is required to execute the direct differentiation method. To summarize:

- Rayleigh damping with the *initial* stiffness requires the material to produce the initial stiffness, K, and the derivative of the stiffness, $\partial K/\partial\theta$. However, in linear dynamic analysis, that stiffness derivative is required regardless of whether Rayleigh damping is specified.
- Rayleigh damping with the *current* stiffness in nonlinear dynamic analysis requires the material to produce the conditional derivative, $\partial K/\partial\theta\,|_{\varepsilon \text{ fixed}}$, as well as the unique stiffness derivative $\partial K/\partial\varepsilon$.

The first item is straightforward; for the Bouc-Wen model the initial stiffness is E and the derivative of the initial stiffness is either zero or unity, depending on whether θ is E. The second item, related to the use of the current stiffness, requires further attention. In the document on the Bouc-Wen model, two approaches are outlined for obtaining the tangent stiffness. Only one yields accurate results for the sought derivative; namely, the one that differentiates Eq. (5). From the document on the Bouc-Wen model, an equation containing $dz_{n+1}/d\varepsilon_{n+1}$ on both sides of the equal sign is presented. That equation is first differentiated through with respect to θ, and separately with respect ε_{n+1} in order. The resulting equations are solved for $d(dz_{n+1}/d\varepsilon_{n+1})/d\theta$ and $d(dz_{n+1}/d\varepsilon_{n+1})/d\varepsilon_{n+1}$, i.e., $dK/d\theta$ and $dK/d\varepsilon_{n+1}$, respectively, in order to obtain the two stiffness derivatives sought when the Rayleigh damping matrix contains the current tangent stiffness. Those equations appear in the G2 code posted on this website.