Adjoint Method

An illustration of the adjoint method is made by considering the limit-state function

$$g = u_o - u(\theta) \tag{1}$$

where u_o=scalar threshold, u=selected displacement from the response vector $\mathbf{u}$, and θ=input parameter to the finite element model. In order to conduct certain types of reliability analysis, we require the derivative

$$\frac{\partial g}{\partial \theta} = \frac{\partial g}{\partial \mathbf{u}} \cdot \frac{\partial \mathbf{u}}{\partial \theta} \tag{2}$$

For this illustration, consider linear static analysis, in which response sensitivities are governed by the equation

$$\frac{\partial \mathbf{u}}{\partial \theta} = \mathbf{K}^{-1}\left(\frac{\partial \mathbf{F}}{\partial \theta} - \frac{\partial \mathbf{K}}{\partial \theta}\mathbf{u}\right) \tag{3}$$

Substitution of Eq. (3) into Eq. (2) yields

$$\frac{\partial g}{\partial \theta} = \underbrace{\frac{\partial g}{\partial \mathbf{u}} \cdot \mathbf{K}^{-1}}_{\boldsymbol{\lambda}}\left(\frac{\partial \mathbf{F}}{\partial \theta} - \frac{\partial \mathbf{K}}{\partial \theta}\mathbf{u}\right) \tag{4}$$

where the vector $\boldsymbol{\lambda}$ is identified. Examination of Eq. (4) suggests that $\boldsymbol{\lambda}$ is determined from the linear system of equations

$$\mathbf{K}\boldsymbol{\lambda} = \frac{\partial g}{\partial \mathbf{u}} \tag{5}$$

Notice that $\partial g/\partial \mathbf{u}$ is a vector of zeros except one component equal to minus unity. Once $\boldsymbol{\lambda}$ is determined, Eq. (4) implies that the sought derivative of the limit-state function is determined by the dot product

$$\frac{\partial g}{\partial \theta} = \boldsymbol{\lambda}^T\left(\frac{\partial \mathbf{F}}{\partial \theta} - \frac{\partial \mathbf{K}}{\partial \theta}\mathbf{u}\right) \tag{6}$$

Importantly, $\boldsymbol{\lambda}$ does not vary with θ. That means the vector $\boldsymbol{\lambda}$ is determined once and for all θ. Once a right-hand side is established for a parameter θ, only a computationally inexpensive dot product is required in order to obtain the sought derivative $\partial g/\partial \theta$. This reduces the computational cost of the direct differentiation method because it circumvents the solution of a linear system of equations for every θ. However, for inelastic problems, containing material models with history variables, the full vector $\partial \mathbf{u}/\partial \theta$ is required in “Phase 2” of the sensitivity calculations, and the adjoint method loses its appeal. The adjoint method is implemented in the programming language Python in the linear static G2 code posted on this website.