

Chevalier de Mere's First Problem

This problem was posed by Chevalier de Mere (Antoine Gombaud) to Blaise Pascal, who solved it in communication with Pierre de Fermat in 1654. Two players agree to play a series that consists of several games, where the “best of N games” wins all the stakes. However, they are interrupted and must divide the stakes in a fair manner between them before finishing.

How should the stakes be divided? The answer gave rise to the concept of probability; the stakes should be divided according to the likelihood of victory had the game been played until the end, assuming 50-50 chance of winning each game.

For this example, suppose they play best of seven games, like in playoff hockey, and they are interrupted when Player A has won 3 games and Player B has won 1 game. Letting a and b denote the number of NEEDED wins by Player A and B to clinch the series we have:

$$a = 1;$$

$$b = 3;$$

Pascal's triangle

Pascal's approach involved counting possible scenarios by a technique that became known as Pascal's triangle. This is essentially a method for evaluating the “binomial coefficient,” i.e., the number of possible scenarios in which a player would have won the series. For this purpose, understand that the maximum number of games remaining is

$$\text{maxNumberOfRemainingGames} = a + b - 1$$

which yields: 3

Using Pascal's triangle, i.e., the binomial coefficient, we find the following number of scenarios in which Player A wins ONE game:

$$A_{\text{winsOne}} = \text{Binomial}[\text{maxNumberOfRemainingGames}, 1]$$

which yields: 3

... and the following number of scenarios in which Player A wins TWO games (notice we count series scenarios that would not have been played to Game 7):

$$\text{AwinsTwo} = \text{Binomial}[\text{maxNumberOfRemainingGames}, 2]$$

which yields: 3

... and the following number of scenarios in which Player A wins THREE games (notice we count series scenarios that would not have been played to Game 7):

$$\text{AwinsThree} = \text{Binomial}[\text{maxNumberOfRemainingGames}, 3]$$

which yields: 1

Conversely, the number of series scenarios that would have favoured Player B is only the one where that player wins all three remaining games:

$$\text{BwinsThree} = \text{Binomial}[\text{maxNumberOfRemainingGames}, 3]$$

which yields: 1

Dividing the stakes according to the likelihood of winning the series implies this fraction of the stakes should go to Player A:

$$\frac{\text{AwinsOne} + \text{AwinsTwo} + \text{AwinsThree}}{\text{AwinsOne} + \text{AwinsTwo} + \text{AwinsThree} + \text{BwinsThree}}$$

which yields: $\frac{7}{8}$

... and the fraction that should go to Player B is:

$$\frac{\text{BwinsThree}}{\text{AwinsOne} + \text{AwinsTwo} + \text{AwinsThree} + \text{BwinsThree}}$$

which yields: $\frac{1}{8}$

A slightly different approach

The total number of possible scenarios is:

$$\text{numberOfPossibleScenarios} = 2^{\text{maxNumberOfRemainingGames}}$$

which yields: 8

Directly establish the possible scenarios:

AAA (A wins)
 AAB (A wins)
 ABA (A wins)
 BAA (A wins)
 ABB (A wins)
 BAB (A wins)
 BBA (A wins)
 BBB (B wins)

Notice again that games are included that will not be played because the series is over before “game seven.” Although this was disputed by Roberval and d’Alembert this gives the correct probability. In the list above it is observed that the number of the scenarios that favour player A is:

$$\text{numberOfScenariosFavouringPlayerA} = 7;$$

Hence, the portion of the stakes that should be given to Player A, i.e., the probability that Player A would have won the series, is:

$$\text{ratioOfStakesToPlayerA} = \frac{\text{numberOfScenariosFavouringPlayerA}}{\text{numberOfPossibleScenarios}}$$

which yields: $\frac{7}{8}$

Show the result as percentage:

$$\text{ratioOfStakesToPlayerA} \cdot 100 \text{ // } N$$

which yields: 87.5

Using the union rule with independent events

Let the event E_i denote Player A winning the entire series after Game i , $i=1,2,3$. Assuming statistical independence between games, there are three ways for Player A to win:

1. A wins the first game: $P(E_1) = 0.5$
2. A loses the first game but wins the 2nd one: $P(E_2) = (0.5) (0.5)$
3. A loses the 1st and 2nd game but wins the third: $P(E_3) = (0.5) (0.5) (0.5)$

Because these are mutually exclusive events, the probabilities can be added:

$$P(\text{Player A wins}) = (1/2) + (1/2)(1/2) + (1/2)(1/2)(1/2) = 7/8$$

Verification by Monte Carlo sampling

Set the number of samples:

```
numberOfSamples = 100 000;
```

Initialize the array where continuously updated values of the probability that Player A will win is stored, for plotting purposes:

```
probability = Table[0, {numberOfSamples}, {i, 1, 2}];
```

Before the sampling starts, initialize the variables that contain the number of series wins by each player:

```
seriesWinsByPlayerA = 0;
```

```
seriesWinsByPlayerB = 0;
```

Carry out the Monte Carlo sampling:

```
For[i = 1, i < numberOfSamples + 1, i++,
  gameWinsByPlayerA = 3;
  gameWinsByPlayerB = 1;
  While[gameWinsByPlayerA < 4 && gameWinsByPlayerB < 4,
    sampledWinner = RandomInteger[];
    If[sampledWinner == 1, gameWinsByPlayerA++, gameWinsByPlayerB++];
  ];
  If[gameWinsByPlayerA == 4, seriesWinsByPlayerA++,
    seriesWinsByPlayerB++];
  probability[[i, 1]] = i;
  probability[[i, 2]] =  $\frac{\text{seriesWinsByPlayerA}}{i}$ ;
];
```

Show the result, at the final sample, as a fraction:

```
probability[[numberOfSamples, 2]]
```

which yields: $\frac{43\,731}{50\,000}$

Show the result, at the final sample, as a real number:

```
probability[[numberOfSamples, 2]] // N
```

which yields: 0.87462

To avoid a large PDF file when distributing the solution, plot the probability only at selected samples, i.e., extract probability values from the array with all the probability values:

```
plotFrequency = 10;
```

```
pickedNumbers = Table[i, {i, 1, numberOfSamples, plotFrequency}];
```

```
listToPlot = probability[[pickedNumbers]];
```

Create the plot, including the analytical solution from the previous problem, shown as a thin black line:

```
Show[ListPlot[listToPlot, AxesLabel → {"N", "Probability"},  
  PlotStyle → {Black, Thin}],  
ListLinePlot[{{0, ratioOfStakesToPlayerA},  
  {numberOfSamples, ratioOfStakesToPlayerA}},  
  PlotStyle → {Gray, Thick}]]
```

