Multi-Degree-of-Freedom Dynamics

For reference, the governing equation for linear single-degree-of-freedom (SDOF) problems is the "equation of motion," which reads

$$M \cdot \ddot{u}(t) + C \cdot \dot{u}(t) + K \cdot u(t) = F(t) \tag{1}$$

To arrive at the corresponding equations for multi-degree-of-freedom (MDOF) problems it is not Eq. Page 1 that serves as the starting point. Rather, one considers the "strong form," i.e., differential equation for the boundary value problem at hand, say, a beam in bending. Via the "weak form" and substitution of shape functions, each associated with a DOF, the problem is discretized. Following that approach, the stiffness matrix, **K**, the mass matrix, **M**, and the load vector **F** are derived in other documents on this website, leading to the following linear system of equilibrium equations:

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \mathbf{F} \tag{2}$$

where **u** is the vector of DOFs and each dot above **u** means one differentiation with respect to time.

Damping

While inertia forces are functions of acceleration, and elastic forces are functions of displacement, damping forces are functions of velocity. Sometimes called viscous forces, they are represented by the second term of Eq. Page 1. In SDOF dynamic, damping is often specified as the damping ratio, ξ, defined as

$$\xi = \frac{C}{2 \cdot \sqrt{K \cdot M}} \tag{3}$$

where C is the damping coefficient in Eq. Page 1. Damping ratios around 3% to 5% are often assumed for structural systems. Now consider the following system of equilibrium equations for MDOF problems:

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{C}\dot{\mathbf{u}} + \tilde{\mathbf{F}}(\mathbf{u}, \dot{\mathbf{u}}) = \mathbf{F} \tag{4}$$

where $\tilde{\mathbf{F}}(\mathbf{u}, \dot{\mathbf{u}}) = \mathbf{Ku}$ for linear problems. The velocity enters in the second and third terms, presenting two venues for introducing damping. Formulation of the internal resisting forces, i.e., the material models, to include velocity-dependent forces in $\tilde{\mathbf{F}}(\mathbf{u}, \dot{\mathbf{u}})$ is possible but not the most common approach. Rather, artificial damping is introduced by creating some damping matrix, **C**. The most common form of **C** is Rayleigh damping, which is written as a linear combination of the mass and stiffness matrix (Chopra 1995):

$$\mathbf{C} = a \cdot \mathbf{M} + b \cdot \mathbf{K} \tag{5}$$

The two constants a and b can be determined to target the damping ratio ξ at the frequencies ω_1 and ω_2 (Chopra 1995):

$$a = \omega_1 \cdot \omega_2 \cdot \left(\frac{2 \cdot \xi}{\omega_1 + \omega_2} \right) \tag{6}$$

$$b = \left(\frac{2 \cdot \xi}{\omega_1 + \omega_2} \right) \tag{7}$$

but caution must be exercised because the damping at other frequencies can be dramatically different from the target damping ratios. For this reason, it is prudent to plot the function

$$\xi(\omega) = \frac{a}{2 \cdot \omega} + \frac{b \cdot \omega}{2} \tag{8}$$

to view the distribution of the damping on different frequencies.

Eigenvalue Analysis

Undamped SDOF problems are found to have the natural frequency of vibration

$$\omega_n = \sqrt{\frac{k}{m}} \tag{9}$$

MDOF problems have as many natural frequencies as the number of DOFs. They are determined by first defining the trial solution

$$\mathbf{u}(t) = \mathbf{u}_o \cdot \sin(\omega \cdot t) \tag{10}$$

Substitution of Eq. (10) into

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \mathbf{0} \tag{11}$$

yields the eigenvalue problem

$$\left(\mathbf{K} - \omega^2 \cdot \mathbf{M} \right) \mathbf{u}_o = \mathbf{0} \tag{12}$$

where the eigenvalues, i.e., natural frequencies of vibration are determined from

$$\det\left(\mathbf{K} - \omega^2 \cdot \mathbf{M} \right) = 0 \tag{13}$$

and the corresponding eigenmodes, i.e., shapes of the structure in each mode are determined by setting one component of $\mathbf{u}_o$ equal to unity and solving for the others.

Modal Analysis

Suppose a structure has N DOFs and that the eigenmode for mode number n from the previous section, normalized to have unit norm, corresponding to the natural frequency ω_n, is denoted $\boldsymbol{\phi}_n$. If those modes are collected in the matrix $\boldsymbol{\Phi}$ and we set $\mathbf{u}=\boldsymbol{\Phi}\mathbf{q}$, where $\mathbf{q}$ are generalized coordinates, as mentioned in the document on Energy Methods, then the linear version of Eq. (4) without external loads reads

$$\mathbf{M}\boldsymbol{\Phi}\ddot{\mathbf{q}} + \mathbf{C}\boldsymbol{\Phi}\dot{\mathbf{q}} + \mathbf{K}\boldsymbol{\Phi}\mathbf{q} = \mathbf{F} \tag{14}$$

Pre-multiplying that equation with $\boldsymbol{\Phi}^{\mathrm{T}}$ yields

$$\left[\mathbf{\Phi}^T\mathbf{M}\mathbf{\Phi}\right]\ddot{\mathbf{q}}+\left[\mathbf{\Phi}^T\mathbf{C}\mathbf{\Phi}\right]\dot{\mathbf{q}}+\left[\mathbf{\Phi}^T\mathbf{K}\mathbf{\Phi}\right]\mathbf{q}=\mathbf{\Phi}^T\mathbf{F} \tag{15}$$

It turns out that all the matrices in square brackets are diagonal matrices, as long as **C** is defined as earlier by Rayleigh damping. In other words, the mode shapes *decouple* the system of equations into SDOF problems. This decoupling is referred to as modal analysis. Each SDOF problem is obtained by picking values from the diagonalized matrices in Eq. (15) or in the following manner. Consider the eigenvector $\boldsymbol{\phi}_n$, normalized to unit norm. Now consider the special case $\mathbf{u}=\boldsymbol{\phi}_n \cdot q_n$. With that displacement vector, the linear version of Eq. (4), without external loads reads

$$\mathbf{M}\boldsymbol{\phi}_n\cdot\ddot{q}_n+\mathbf{C}\boldsymbol{\phi}_n\cdot\dot{q}_n+\mathbf{K}\boldsymbol{\phi}_n\cdot q_n=\mathbf{F} \tag{16}$$

Pre-multiplying that equation with the transpose of $\boldsymbol{\phi}_n$ yields

$$\underbrace{\left(\boldsymbol{\phi}_n^T\mathbf{M}\boldsymbol{\phi}_n\right)}_{M_n}\cdot\ddot{q}_n+\underbrace{\left(\boldsymbol{\phi}_n^T\mathbf{C}\boldsymbol{\phi}_n\right)}_{C_n}\cdot\dot{q}_n+\underbrace{\left(\boldsymbol{\phi}_n^T\mathbf{K}\boldsymbol{\phi}_n\right)}_{K_n}\cdot q_n=\underbrace{\boldsymbol{\phi}_n^T\mathbf{F}}_{F_n} \tag{17}$$

where the M_n=modal mass, C_n=modal damping, K_n=modal stiffness, and F_n=modal force for vibration mode number n. The modal decoupling that is described in this section is, for linear problems and Rayleigh damping, an alternative to the time-stepping methods covered in another document on this webpage.

Ground Motion

When ground motions are applied to the structure then the force vector is

$$\mathbf{F}=-\mathbf{M}\cdot\mathbf{\Gamma}\cdot\begin{Bmatrix}\ddot{u}_{gx}\\ \ddot{u}_{gy}\\ \ddot{u}_{gz}\end{Bmatrix} \tag{18}$$

where $\ddot{u}_{gi}$ is the acceleration in the i-direction and $\mathbf{\Gamma}$ is a matrix with as many columns as there are ground motion directions. Essentially, $\mathbf{\Gamma}$ assigns ground motion accelerations to the appropriate degrees of freedom. For example, for a 3D ground motion applied to the cantilevered column shown in Figure 1, the force vector is:

$$\mathbf{F}=-\mathbf{M}\cdot\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\\0&0&0\\0&0&0\\0&0&0\end{bmatrix}\cdot\begin{Bmatrix}\ddot{u}_{gx}\\ \ddot{u}_{gy}\\ \ddot{u}_{gz}\end{Bmatrix} \tag{19}$$

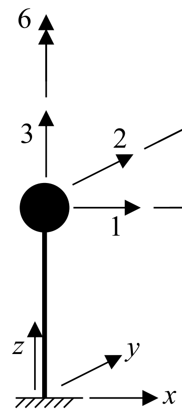

Figure 1: Cantilever with mass.

References

Chopra, A. K. (1995). *Dynamics of Structures: Theory and Applications to Earthquake Engineering*. Prentice Hall.