A short course on

Probabilities and Random Variables

This video:

Classical & Bayesian Statistical Inference

Terje's Toolbox is freely available at terje.civil.ubc.ca
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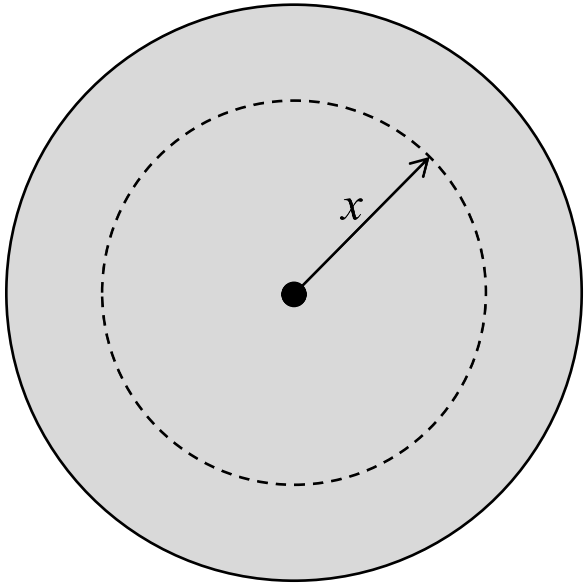
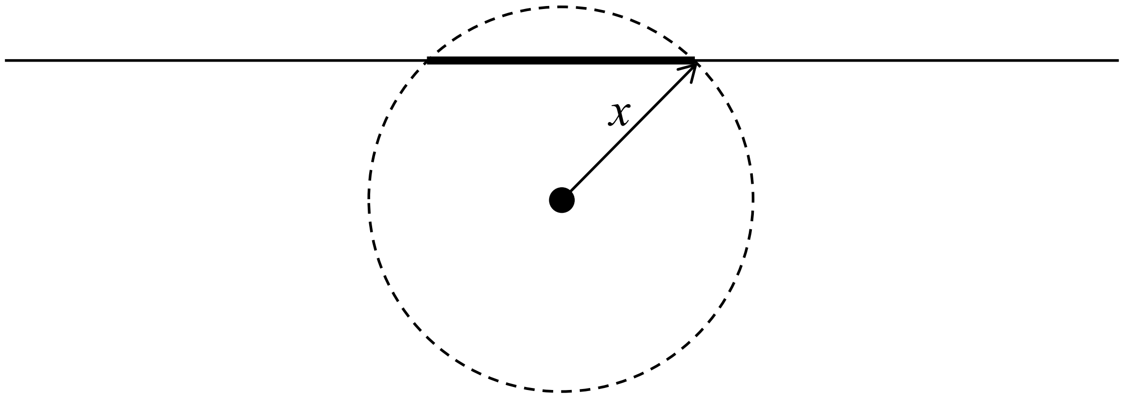
Statistical Inference

Geometrical	Classical	Bayesian
Compare lengths, areas, volumes	Averaging observed data	Consider model parameters as random variables

Geometry

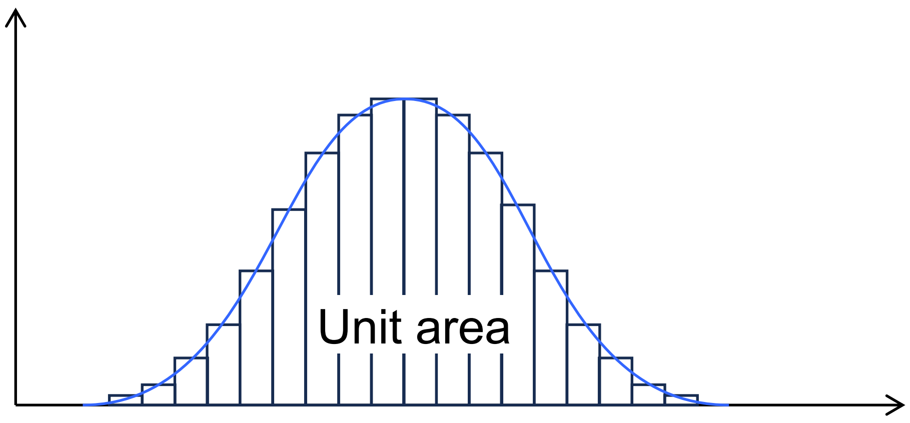


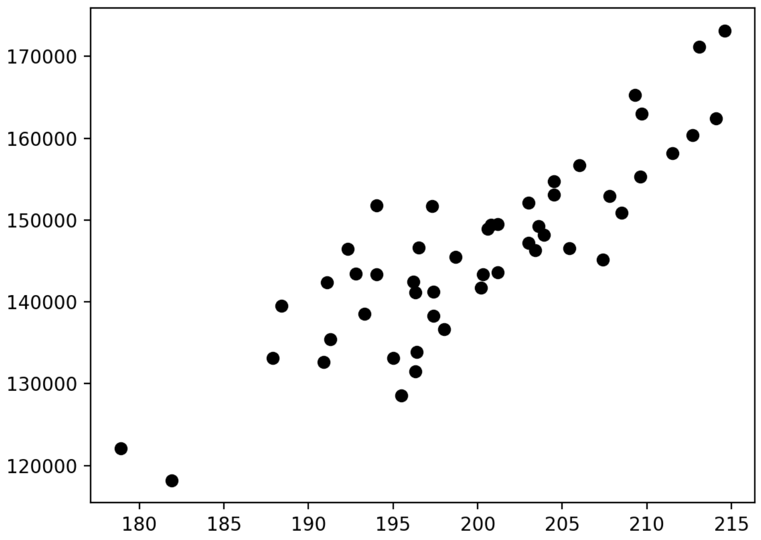
$$F(x) = P(X \le x) = \left\{ \begin{array}{l} \left(\dfrac{L_{X \le x}}{L_{total}} \right) \\ \left(\dfrac{A_{X \le x}}{A_{total}} \right) \\ \left(\dfrac{V_{X \le x}}{V_{total}} \right) \end{array} \right.$$

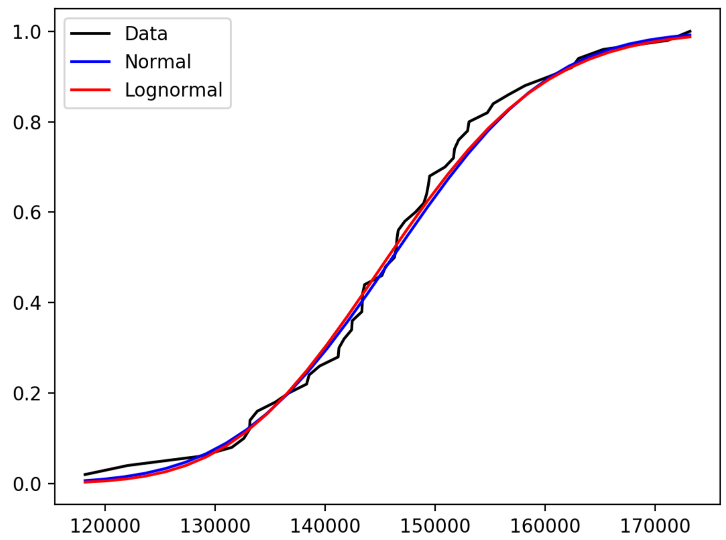


Diagrams

- Histogram
- Frequency diagram
- Cumulative frequency diagram
- Scatter diagram







Mean

Average, expectation

$$\overline{x} = \frac{1}{n} \cdot \sum_{i=1}^{n} x_i$$

Standard Deviation

Basic expression:

$$s^2 = \frac{1}{n-1} \cdot \sum_{i=1}^{n} (x_i - \overline{x})^2$$

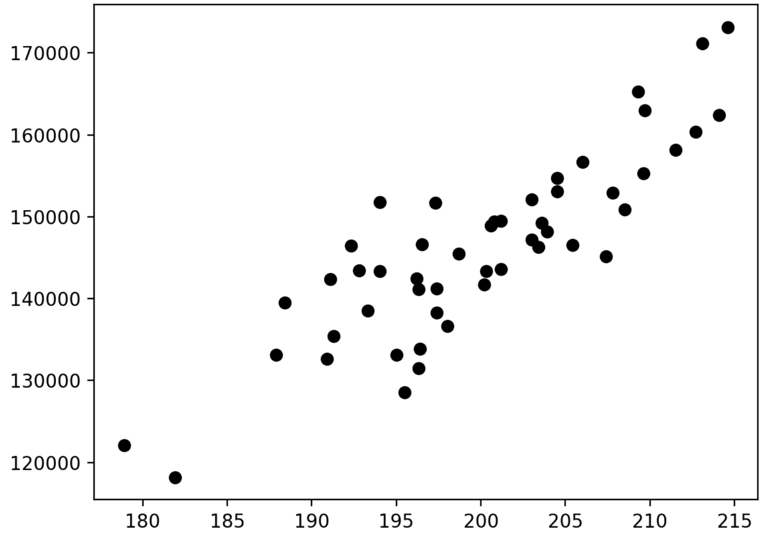
n–1 for the expectation of s to match σ

Computationally more efficient expression:

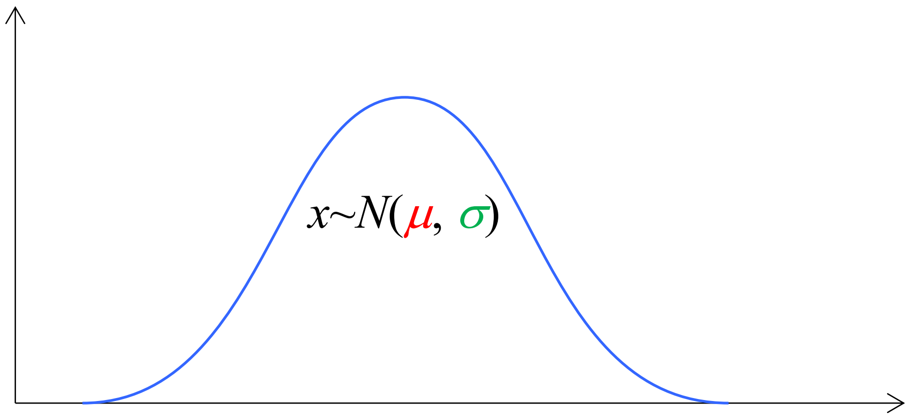
$$\begin{aligned} s^2 &= \frac{1}{n-1} \cdot \sum_{i=1}^{n} (x_i - \overline{x})^2 \\ &= \frac{1}{n-1} \cdot \left(\sum_{i=1}^{n} x_i^2 + \sum_{i=1}^{n} \overline{x}^2 - \sum_{i=1}^{n} 2 x_i \overline{x} \right) \\ &= \frac{1}{n-1} \cdot \left(\left(\sum_{i=1}^{n} x_i^2 \right) + n\overline{x}^2 - 2n\overline{x}^2 \right) \\ &= \frac{1}{n-1} \cdot \left(\left(\sum_{i=1}^{n} x_i^2 \right) - n \cdot \overline{x}^2 \right) \end{aligned}$$

Correlation Coefficient

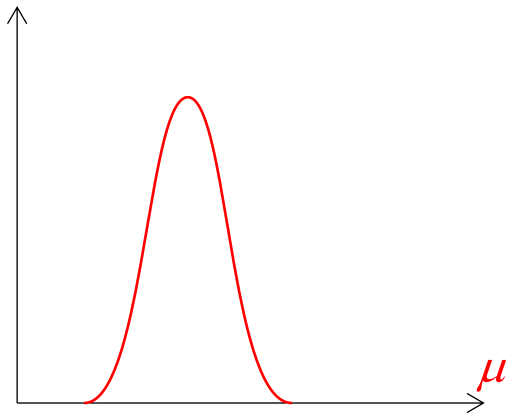
$$\rho = \frac{1}{n-1} \cdot \left(\frac{\sum_{i=1}^{n} x_i y_i - n \cdot \bar{x} \cdot \bar{y}}{s_x \cdot s_y} \right)$$

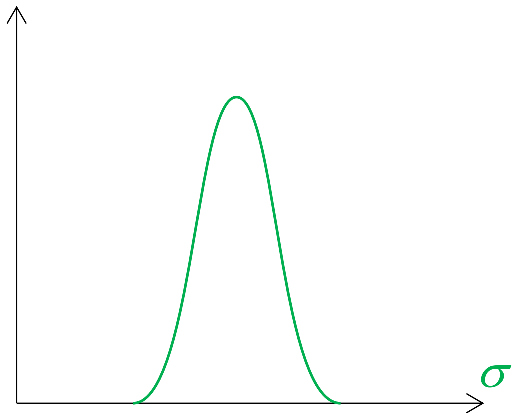


Bayesian Updating



$$P(E_1 \mid E_2) = \frac{P(E_2 \mid E_1)}{P(E_2)} \cdot P(E_1)$$





$$f''(\theta) = \frac{L(\theta)}{c} \cdot f'(\theta)$$

$$c = \int_{-\infty}^{\infty} L(\theta) \cdot f'(\theta) d\theta$$

Likelihood Function

$$f''(\theta) = \frac{L(\theta)}{c} \cdot f'(\theta)$$

"Probability of observing the observation"

Suppose we have observed $X=x$

$L(\theta) \propto \mathrm{P}(X=x \mid \theta) \propto f(x,\theta)$, where θ is, for example, the mean

$\propto p(x,\theta)$, where θ is, for example, the failure probability in Bernoulli trials

Suppose we have observed $X<x$

$L(\theta) \propto \mathrm{P}(X<x \mid \theta) = F(x,\theta)$

Prior

$$f''(\theta) = \frac{L(\theta)}{c} \cdot f'(\theta)$$

Previous posterior

Uniform and non-informative prior

Conjugate prior:

$f''(\theta)$ becomes same type as $f'(\theta)$

Leads to updating rules

Updating Rules for Conjugate Priors

Random variable	Observation	Conjugate prior	Updating rule
X~Binomial(p,n)	x occurrences in n trials	p~Beta(a,b)	$a''=a'+x$ $b''=b'+n-x$
X~Geometric(p)	x trials until first occurrence	p~Beta(a,b)	$a''=a'+1$ $b''=b'+x-1$
X~NegativeBinomial(p,k)	x trials to k^{th} occurrence	p~Beta(a,b)	$a''=a'+k$ $b''=b'+x-k$
X~Poisson(λ,T)	x occurrences in T	λ~Gamma(ν,k)	$k''=k'+x$ $\nu''=\nu'+T$
X~Exp(λ)	n observations of x	λ~Gamma(ν,k)	$k''=k'+n$ $\nu''=\nu'+\Sigma x_i$

More Updating Rules

X~Normal($\underline{\mu}$,$\underline{\sigma}$)	n observations of x	$\underline{\mu}$~Normal(μ,σ)	$\mu'' = \dfrac{\bar{x}\cdot\sigma'^2 + \mu'\left(\dfrac{\underline{\sigma}^2}{n}\right)}{\sigma'^2 + \dfrac{\underline{\sigma}^2}{n}}$ $\sigma'' = \sqrt{\dfrac{\sigma'^2\cdot\left(\dfrac{\underline{\sigma}^2}{n}\right)}{\sigma'^2 + \left(\dfrac{\underline{\sigma}^2}{n}\right)}}$
X~Lognormal(ζ,$\underline{\sigma}$)	n observations of x with average $\bar{x}$	ζ~Normal(μ,σ)	$\mu'' = \dfrac{\overline{\ln(x)}\cdot\sigma'^2 + \mu'\left(\dfrac{\underline{\sigma}^2}{n}\right)}{\sigma'^2 + \dfrac{\underline{\sigma}^2}{n}}$ $\sigma'' = \sqrt{\dfrac{\sigma'^2\cdot\left(\dfrac{\underline{\sigma}^2}{n}\right)}{\sigma'^2 + \left(\dfrac{\underline{\sigma}^2}{n}\right)}}$

Posterior Statistics

$$\mu_\theta = \int_{-\infty}^{\infty} \theta \cdot f''(\theta) d\theta$$

$$\sigma_\theta^2 = \int_{-\infty}^{\infty} (\theta - \mu_\theta)^2 \cdot f''(\theta) d\theta$$

$$\mathbf{M}_\theta = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \boldsymbol{\theta} \cdot f''(\boldsymbol{\theta}) d\boldsymbol{\theta}$$

$$\boldsymbol{\Sigma}_{\theta\theta} = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left(\boldsymbol{\theta} - \mathbf{M}_\theta\right) \cdot \left(\boldsymbol{\theta} - \mathbf{M}_\theta\right)^T \cdot f''(\boldsymbol{\theta}) d\boldsymbol{\theta}$$

Predictive Distribution

Total probability integration:

$$f(x) = \int_{-\infty}^{\infty} f(x|\theta) \cdot f''(\boldsymbol{\theta}) \, d\boldsymbol{\theta}$$

More lectures:

Terje’s Toobox:

terje.civil.ubc.ca