A short course on

Structural Members

This video:

Warping Torsion

Terje's Toolbox is freely available at terje.civil.ubc.ca
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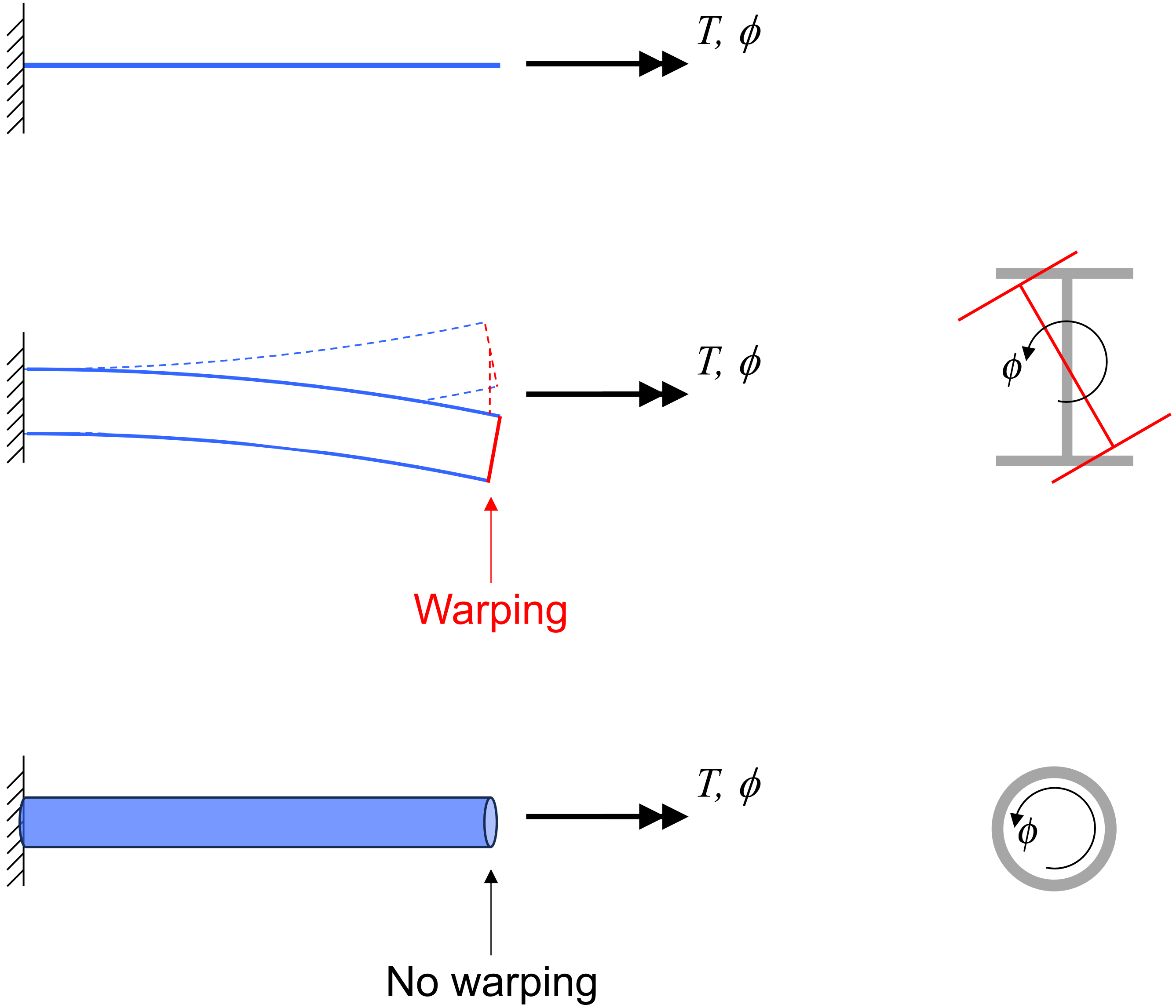
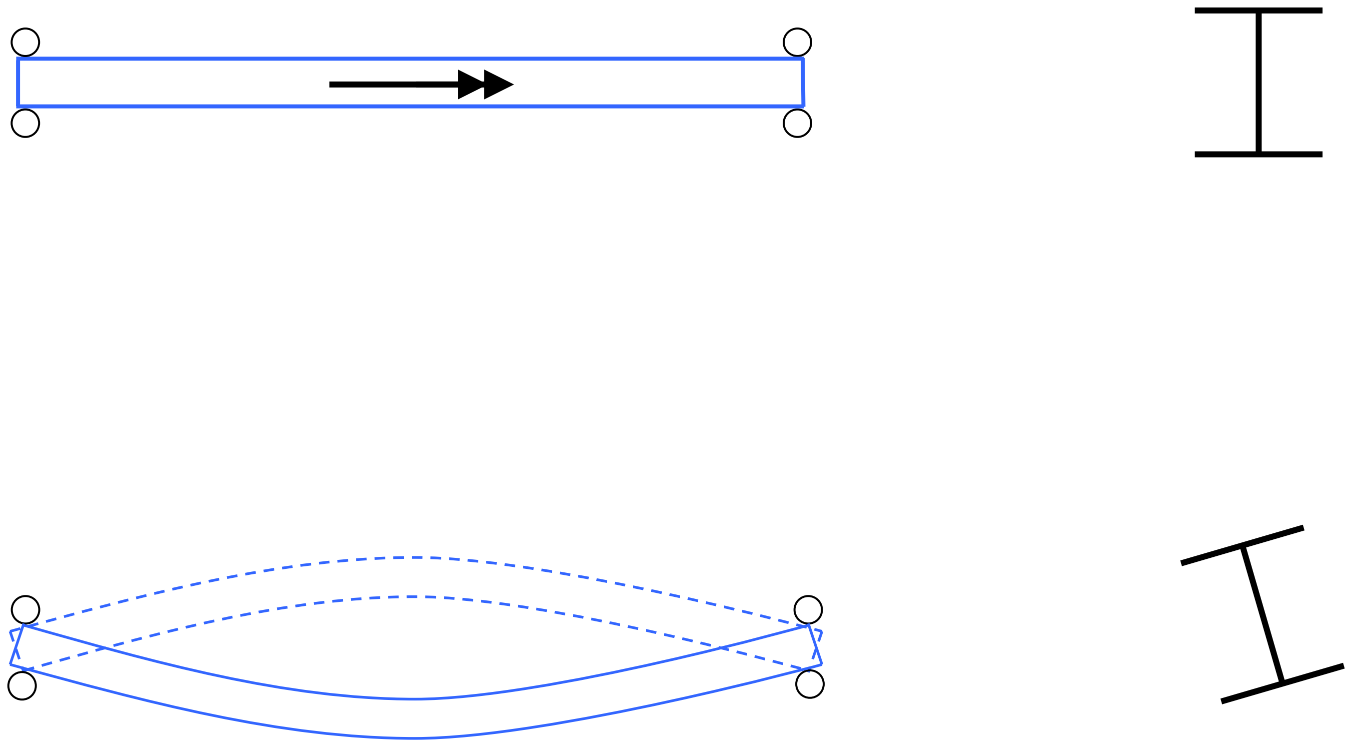
Warping

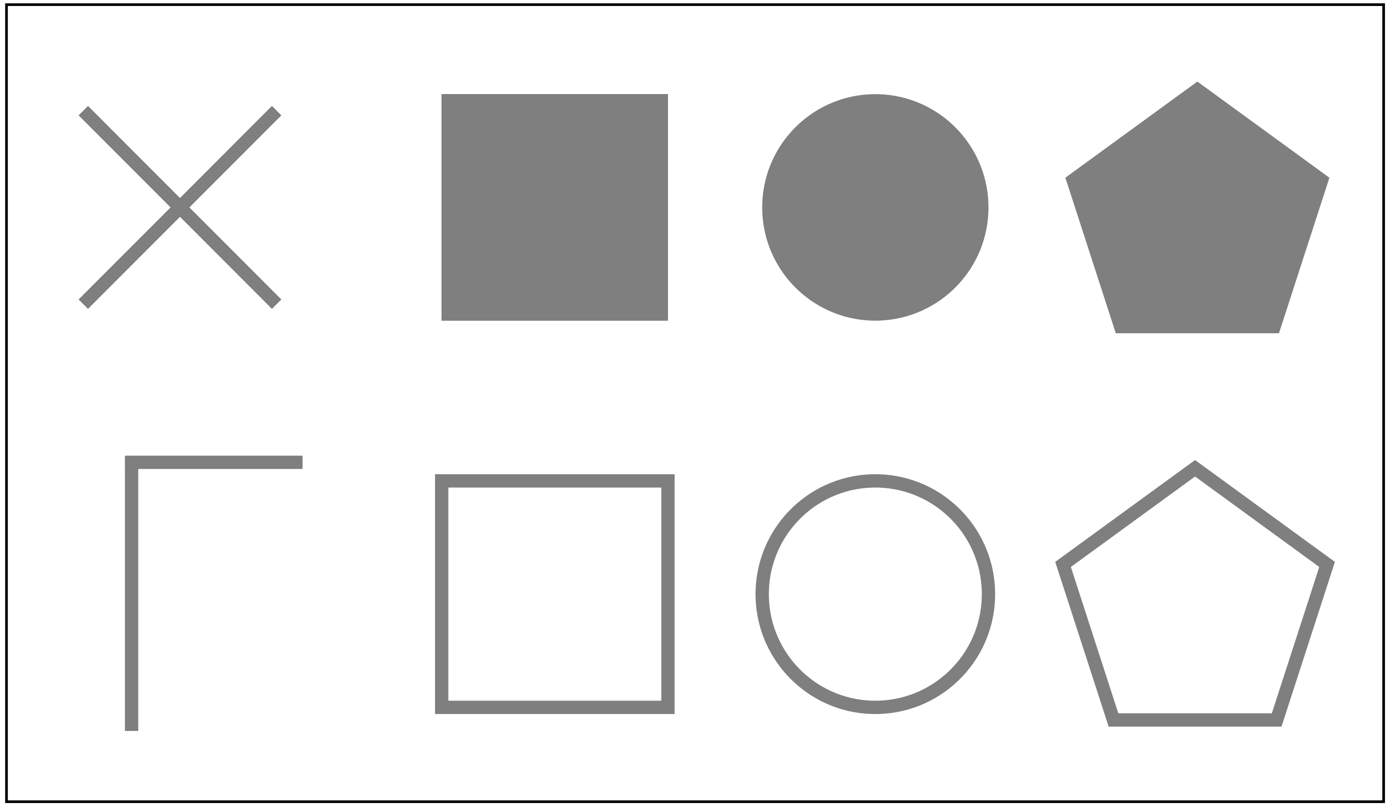


Axial Stress Develops

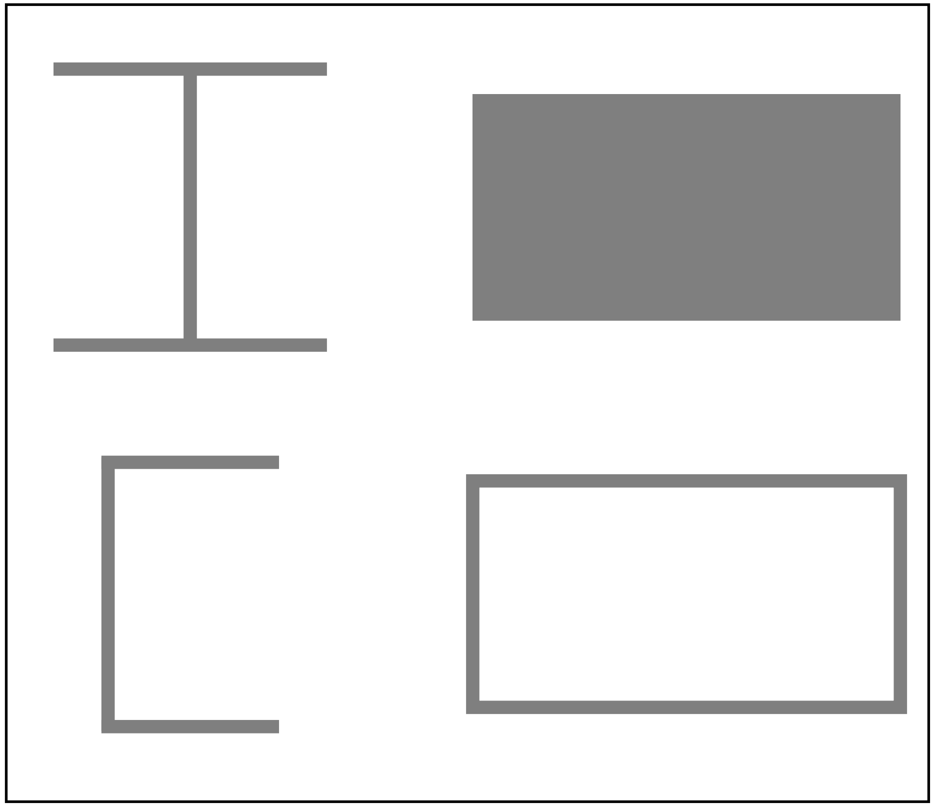


Do they warp?

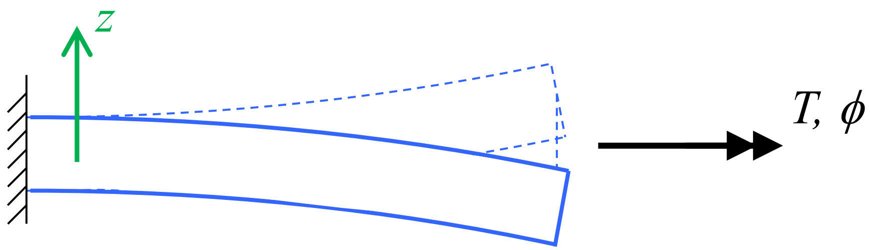
Do NOT warp

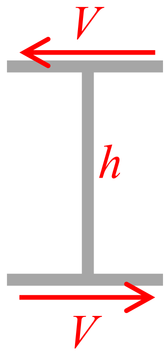


Warp!



Wide-flange Cross-section

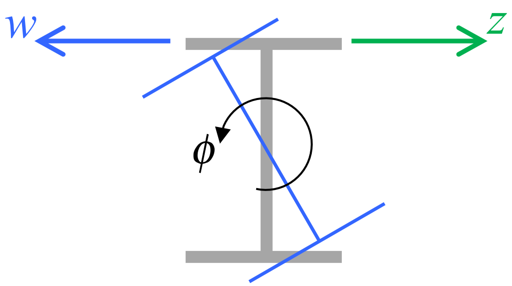




$$T_{warping} = h \cdot V$$

$$V = \frac{dM}{dx} \quad \Rightarrow \quad T_{warping} = h \cdot \frac{dM}{dx}$$

$$M = EI_{flange} \cdot \frac{d^2 w}{dx^2} \quad \Rightarrow \quad T_{warping} = h \cdot EI_{flange} \cdot \frac{d^3 w}{dx^3}$$



$$w = -\phi \cdot \frac{h}{2} \quad \Rightarrow \quad T_{warping} = -\frac{h^2}{2} \cdot EI_{flange} \cdot \frac{d^3 \phi}{dx^3}$$

$$C_w = I_{flange} \cdot \frac{h^2}{2}$$

$$T_{warping} = -EC_w \cdot \frac{d^3 \phi}{dx^3}$$

Saint Venant + Warping Torsion

$$T = GJ \cdot \frac{d\phi}{dx} - EC_w \cdot \frac{d^3\phi}{dx^3}$$

$$m_x = -\frac{dT}{dx}$$

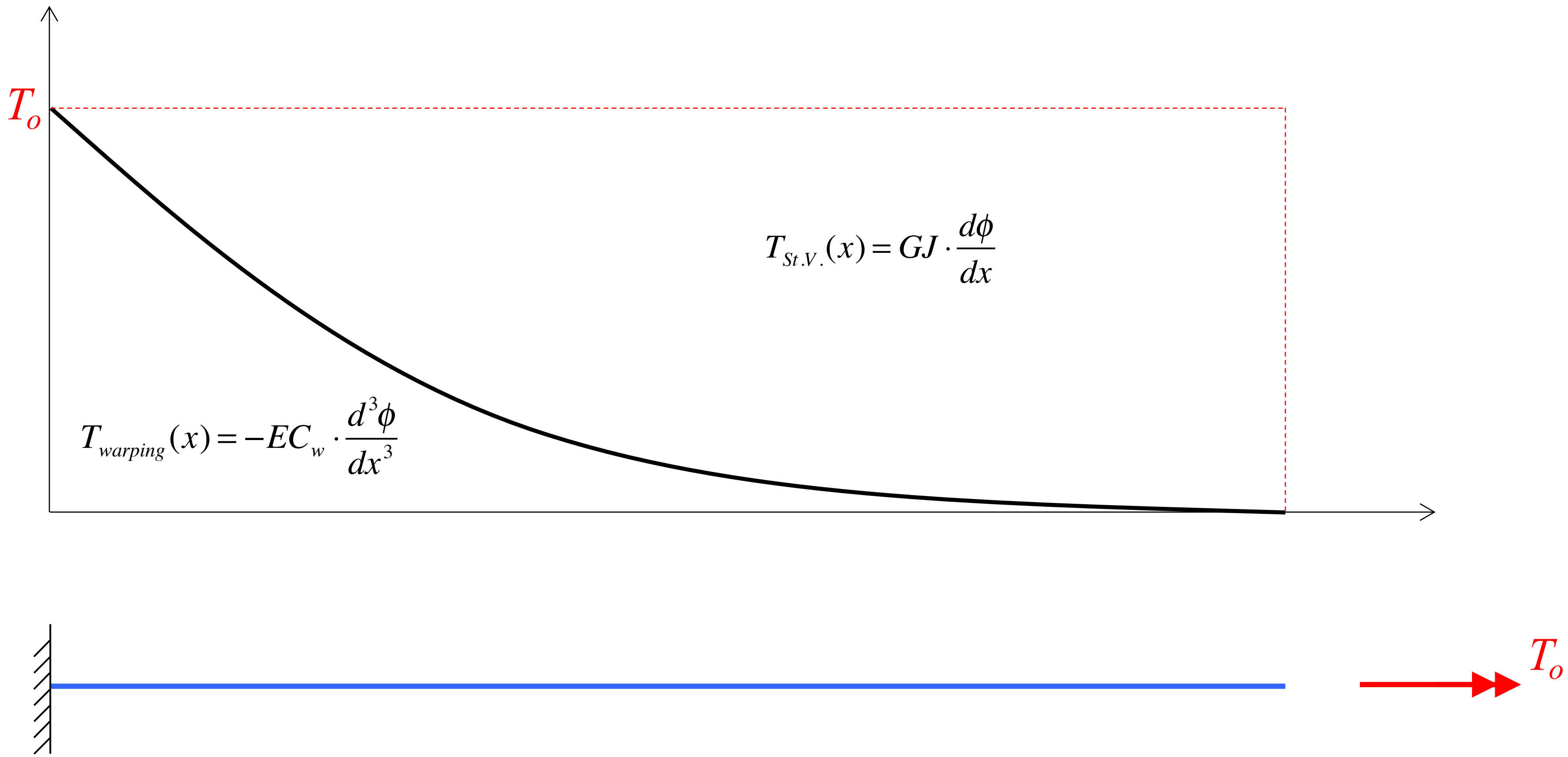
$$EC_w \cdot \frac{d^4\phi}{dx^4} - GJ \cdot \frac{d^2\phi}{dx^2} = m_x$$

$$\gamma^4 - \frac{GJ}{EC_w} \cdot \gamma^2 = 0$$

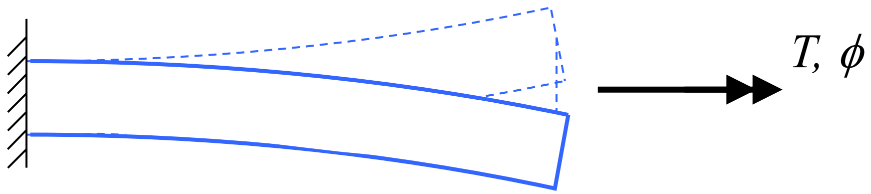
$$\phi(x) = C_1 \cdot \sinh\left(\sqrt{GJ/EC_w} \cdot x\right) + C_2 \cdot \cosh\left(\sqrt{GJ/EC_w} \cdot x\right) + C_3 \cdot x + C_4$$

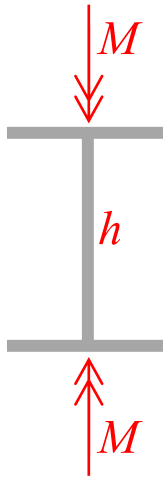
$$\phi(x) = \frac{1}{\sqrt{GJ/EC_w}} \cdot \frac{T_o}{GJ} \cdot \begin{pmatrix} \tanh\left(\sqrt{GJ/EC_w} \cdot L\right) \cdot \left[\cosh\left(\sqrt{GJ/EC_w} \cdot x\right) - 1\right] \\ -\sinh\left(\sqrt{GJ/EC_w} \cdot x\right) + \sqrt{GJ/EC_w} \cdot x \end{pmatrix}$$

How is the torque carried?



Bi-moment





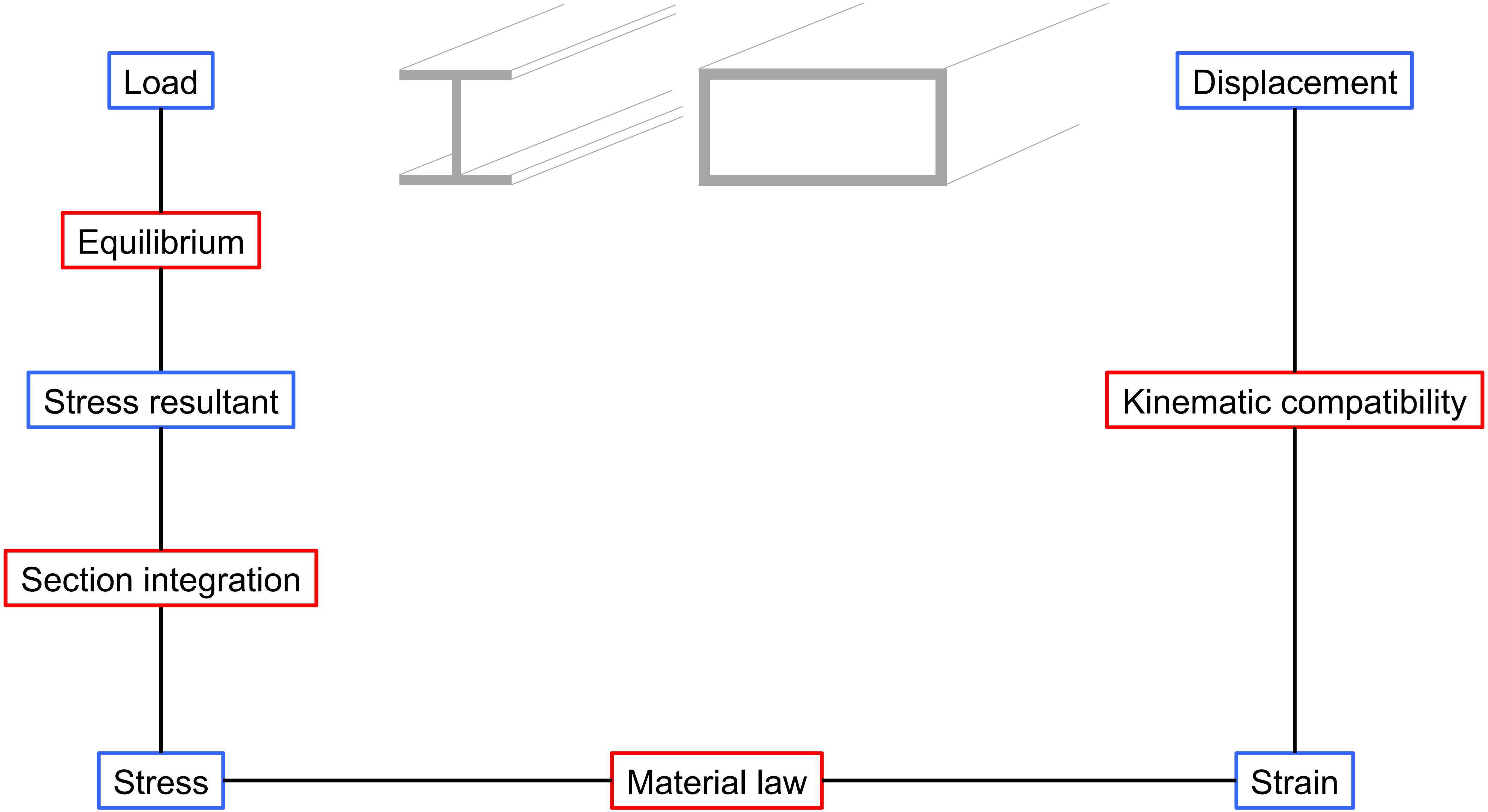
$$B \equiv M \cdot h$$

$$B = EI \cdot \frac{d^2 w}{dx^2} \cdot h$$

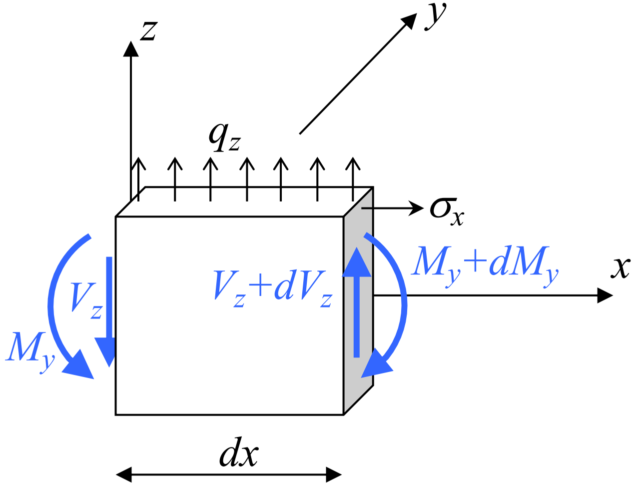
$$B = -EI \cdot \frac{d^2 \phi}{dx^2} \cdot \frac{h^2}{2}$$

$$B = -EC_w \cdot \frac{d^2 \phi}{dx^2}$$

Unified Bending & Torsion

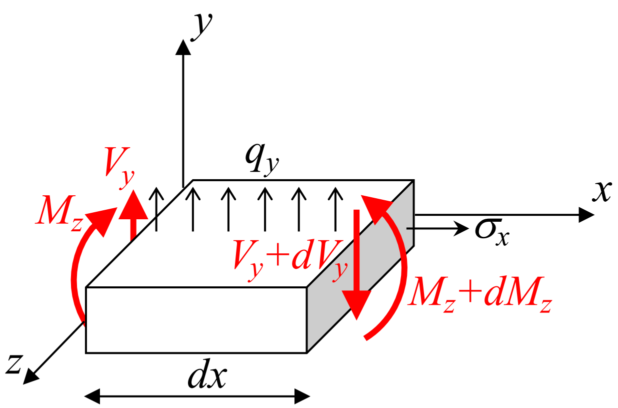


Equilibrium



$$q_z = -\frac{dV_z}{dx}$$

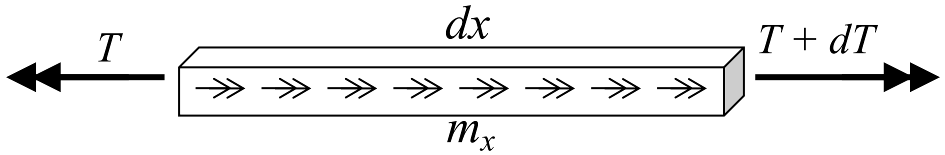
$$V_z = \frac{dM_y}{dx}$$



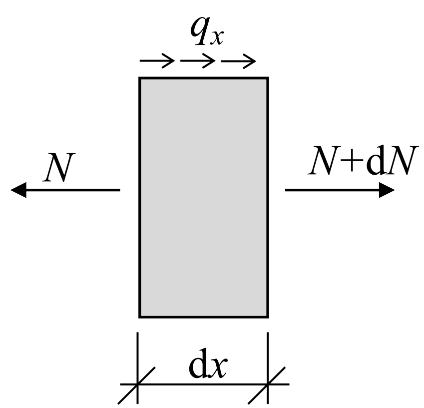
$$q_y = \frac{dV_y}{dx}$$

$$V_y = \frac{dM_z}{dx}$$

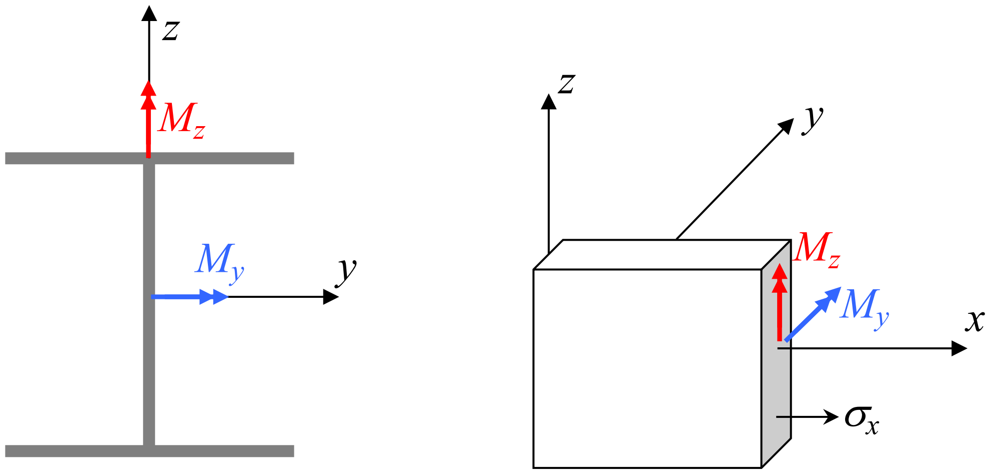
$$m_x = -\frac{dT}{dx}$$



$$q_x = -\frac{dN}{dx}$$



Section Integration



$$N = \int_A \sigma_x \, dA$$

$$M_z = -\int_A \sigma_x \cdot y \, dA$$

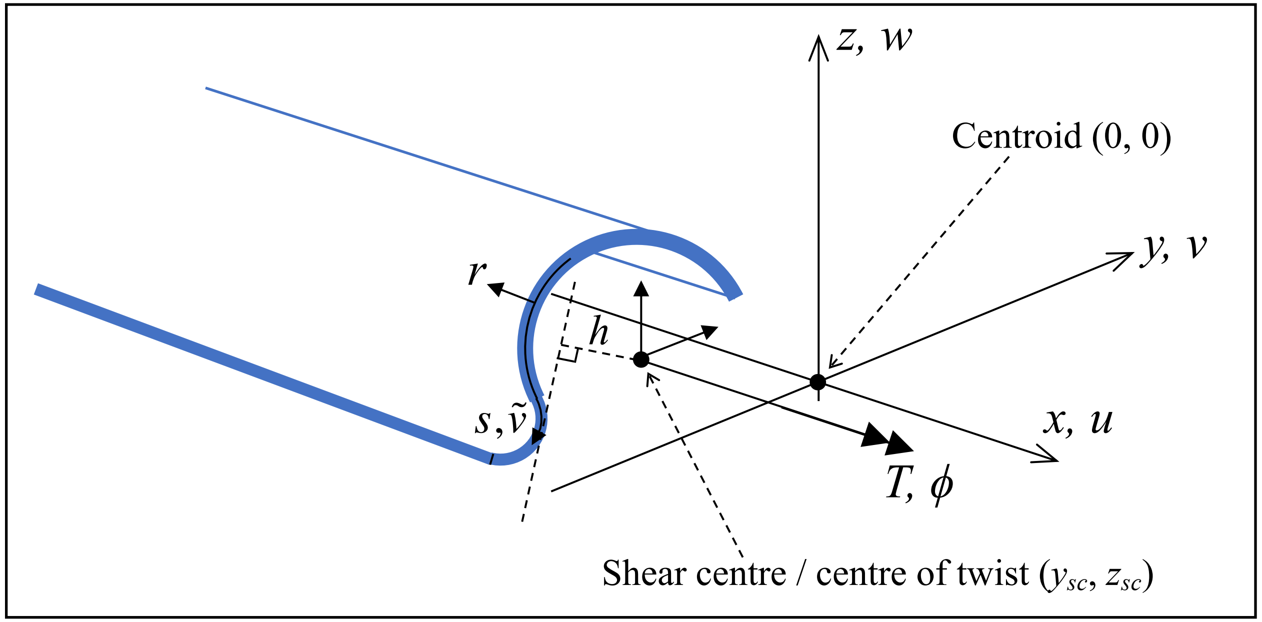
$$M_y = \int_A \sigma_x \cdot z \, dA$$

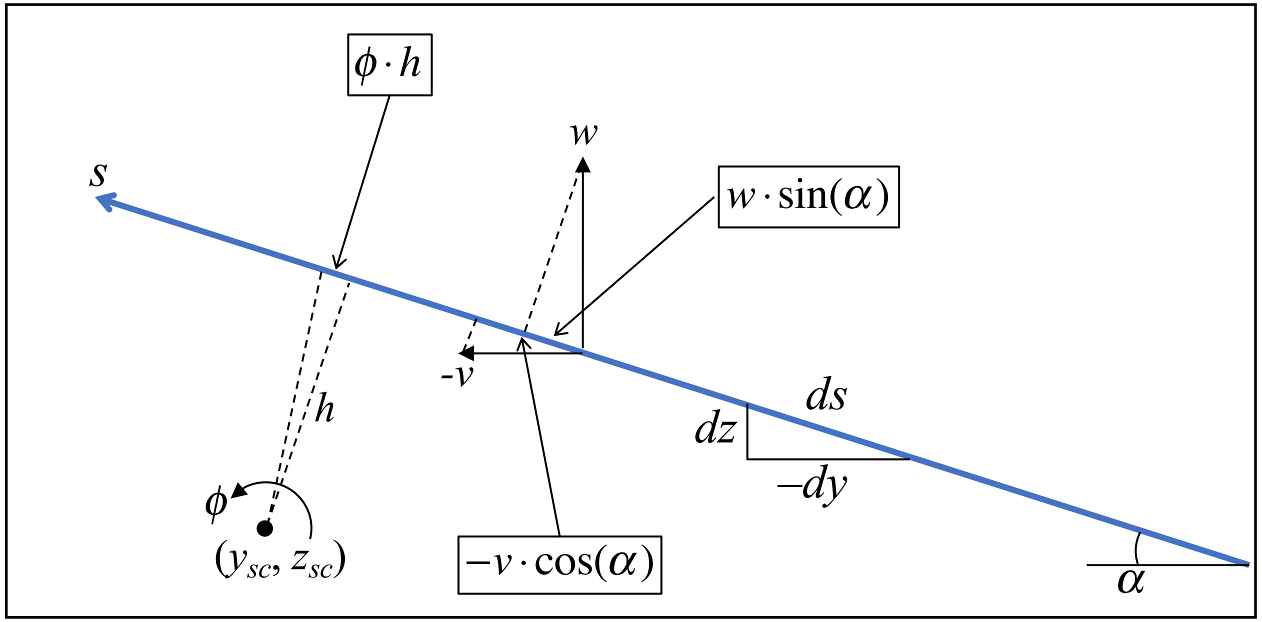
$$B \equiv -\int_A \sigma_x \cdot \Omega \, dA$$

Material Law

$$\sigma_x = E \cdot \varepsilon_x$$

Kinematic Compatibility





$$\tilde{v} = -v \cdot \cos(\alpha) + w \cdot \sin(\alpha) + \phi \cdot h$$

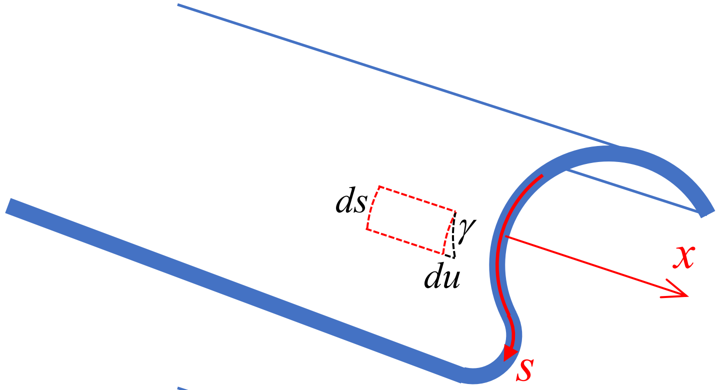
$$\frac{dy}{ds} = -\cos(\alpha)$$

$$\frac{dz}{ds} = \sin(\alpha)$$

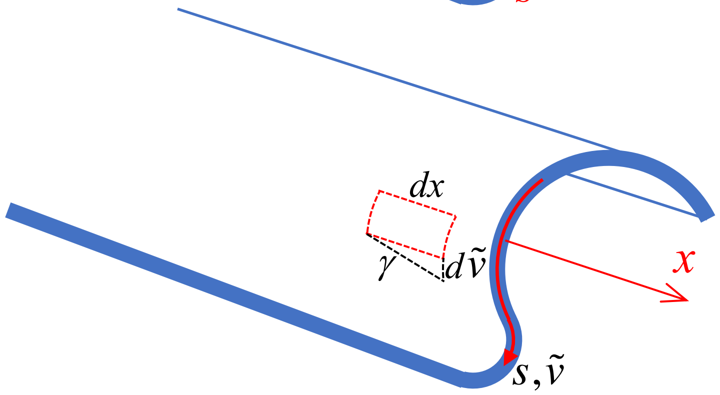
$$\tilde{v} = v \cdot \frac{dy}{ds} + w \cdot \frac{dz}{ds} + \phi \cdot h$$

More Kinematics

$$\varepsilon_x = \frac{du}{dx}$$



$$\gamma_{xs} = \frac{d\tilde{v}}{dx} + \frac{du}{ds} \quad \Rightarrow \quad du = -\frac{\mathrm{d}v}{\mathrm{d}x}\cdot dy - \frac{\mathrm{d}w}{\mathrm{d}x}\cdot dz - \left(\frac{\mathrm{d}\phi}{\mathrm{d}x}\cdot h - \gamma_{xs}\right)\cdot ds$$



$$u(y,z) = u_o - \frac{\mathrm{d}v}{\mathrm{d}x}\cdot y - \frac{\mathrm{d}w}{\mathrm{d}x}\cdot z - \frac{\mathrm{d}\phi}{\mathrm{d}x}\cdot \int h\, ds$$

$$\Omega(s) \equiv \int h\, ds$$

$$\varepsilon_x = \frac{\mathrm{d}u}{\mathrm{d}x} - \frac{\mathrm{d}^2 v}{\mathrm{d}x^2}\cdot y - \frac{\mathrm{d}^2 w}{\mathrm{d}x^2}\cdot z - \frac{\mathrm{d}^2\phi}{\mathrm{d}x^2}\cdot \Omega$$

Closed Cross-section

$$\gamma_{xs} = \frac{\tau_{xs}}{G}$$

$$\tau_{xs} = \frac{K}{t}$$

$$T = 2 \cdot V = 2 \cdot K \cdot A_m$$

$$\gamma_{xs} = \frac{\tau_{xs}}{G} = \frac{K}{G \cdot t} = \frac{T}{G \cdot t \cdot 2 \cdot A_m} = \frac{J}{2 \cdot t \cdot A_m} \cdot \phi'$$

$$\Omega(s) \equiv \int h \, ds$$

$$\Omega(s) = \int \left(h - \frac{J}{2 \cdot t \cdot A_m} \right) ds$$

$\bar{h}$

Differential Equations

$$\sigma_x = E \cdot \frac{\mathrm{d}u}{\mathrm{d}x} - E \cdot \frac{\mathrm{d}^2 v}{\mathrm{d}x^2} \cdot y - E \cdot \frac{\mathrm{d}^2 w}{\mathrm{d}x^2} \cdot z - E \cdot \frac{\mathrm{d}^2 \phi}{\mathrm{d}x^2} \cdot \Omega$$

$$\begin{Bmatrix} N \\ M_z \\ -M_y \\ -B \end{Bmatrix} = E \cdot \begin{bmatrix} \int_A dA & -\int_A y\,dA & -\int_A z\,dA & -\int_A \Omega\,dA \\ -\int_A y\,dA & \int_A y^2\,dA & \int_A y \cdot z\,dA & \int_A y \cdot \Omega\,dA \\ -\int_A z\,dA & \int_A y \cdot z\,dA & \int_A z^2\,dA & \int_A z \cdot \Omega\,dA \\ -\int_A \Omega\,dA & \int_A y \cdot \Omega\,dA & \int_A z \cdot \Omega\,dA & \int_A \Omega^2\,dA \end{bmatrix} \begin{Bmatrix} \frac{du}{dx} \\ \frac{d^2 v}{dx^2} \\ \frac{d^2 w}{dx^2} \\ \frac{d^2 \phi}{dx^2} \end{Bmatrix}$$

$$A = \int_A \mathrm{d}A$$

$$I_z = \int_A y^2\,dA$$

$$I_y = \int_A z^2\,dA$$

$$C_\omega = \int_A \Omega^2\,dA$$

→

$$N = EA \cdot \frac{du}{dx}$$

$$M_z = EI_z \frac{d^2 v}{dx^2}$$

$$M_y = -EI_y \frac{d^2 w}{dx^2}$$

$$B = -EC_w \frac{d^2 \phi}{dx^2}$$

$$q_x = -EA \cdot \frac{d^2 u}{dx^2}$$

$$q_y = EI_z \frac{d^4 v}{dx^4}$$

$$q_z = EI_y \frac{d^4 w}{dx^4}$$

Decoupling Conditions

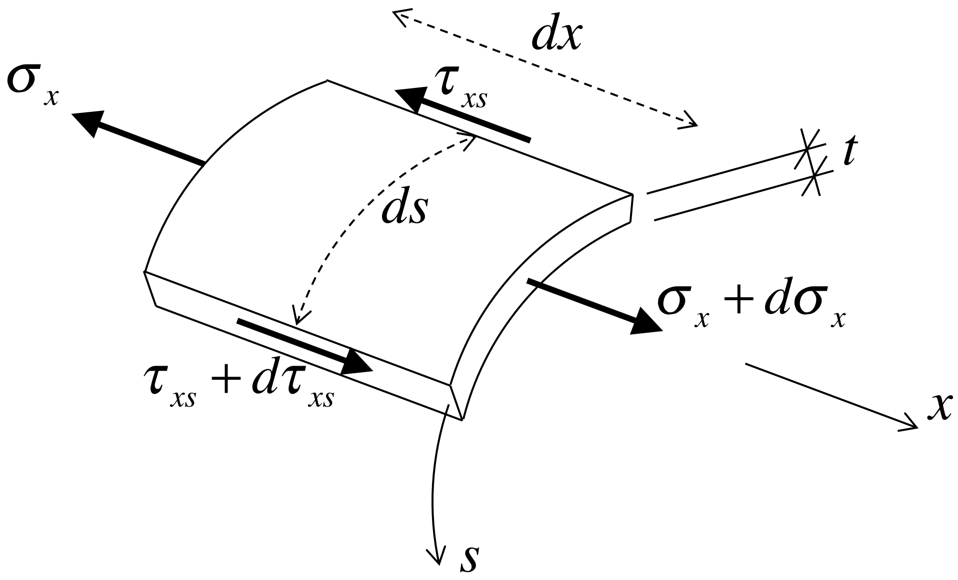
$$\int_A y \; \mathrm{d}A = \int_A z \; \mathrm{d}A = 0$$

$$\int_A y \cdot z \; \mathrm{d}A = 0$$

$$\int_A \Omega \; \mathrm{d}A = 0$$

$$\int_A y \cdot \Omega \; \mathrm{d}A = \int_A z \cdot \Omega \; \mathrm{d}A = 0$$

Shear Flow & Torque



$$d\sigma_x \cdot ds \cdot t + d\tau_{xs} \cdot dx \cdot t = 0 \quad \Rightarrow \quad \frac{d\sigma_x}{dx} \cdot t + \frac{d\tau_{xs}}{ds} \cdot t = 0 \quad \Rightarrow \quad \frac{dq_s}{ds} = -\frac{d\sigma_x}{dx} \cdot t$$

$$T = \int_A \tau_{xs} \cdot t \cdot h \, dA = \int_A q_s \cdot h \, ds = \int_A q_s \, d\Omega = \left[q_s \cdot \Omega \right]_\Gamma - \int_A \Omega \, dq_s$$

$$T = -\int_A \Omega \, dq_s = \int_A \Omega \cdot \frac{d\sigma_x}{dx} \cdot t \, ds = \int_A \Omega \cdot \frac{d\sigma_x}{dx} dA = \frac{d}{dx} \int_A \Omega \cdot \sigma_x \, dA = -\frac{dB}{dx}$$

$$T = GJ \cdot \frac{d\phi}{dx} - EC_w \frac{d^3\phi}{dx^3}$$

$$\boxed{m_x = EC_w \frac{d^4\phi}{dx^4} - GJ \cdot \frac{d^2\phi}{dx^2}}$$

Cross-section Analysis

Omega diagram, Ω

Cross-section constant, C_w

Axial stress, σ

Shear stress, τ

More lectures:

Terje’s Toobox:

terje.civil.ubc.ca