A short course on

Structural Members

This video:

St. Venant Torsion

Terje's Toolbox is freely available at terje.civil.ubc.ca
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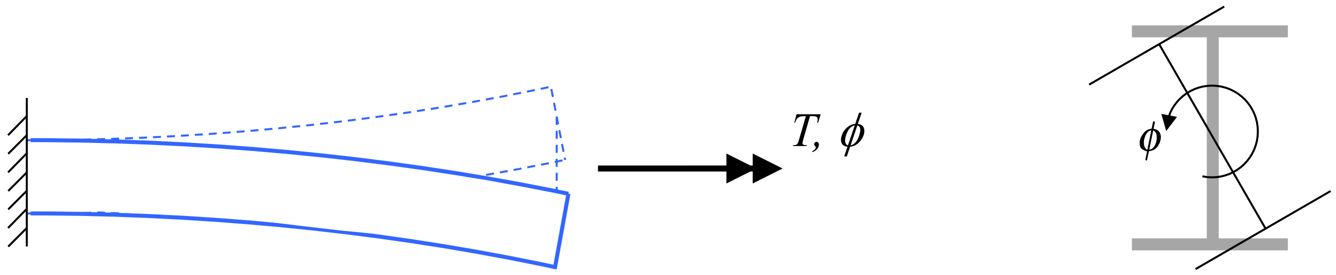
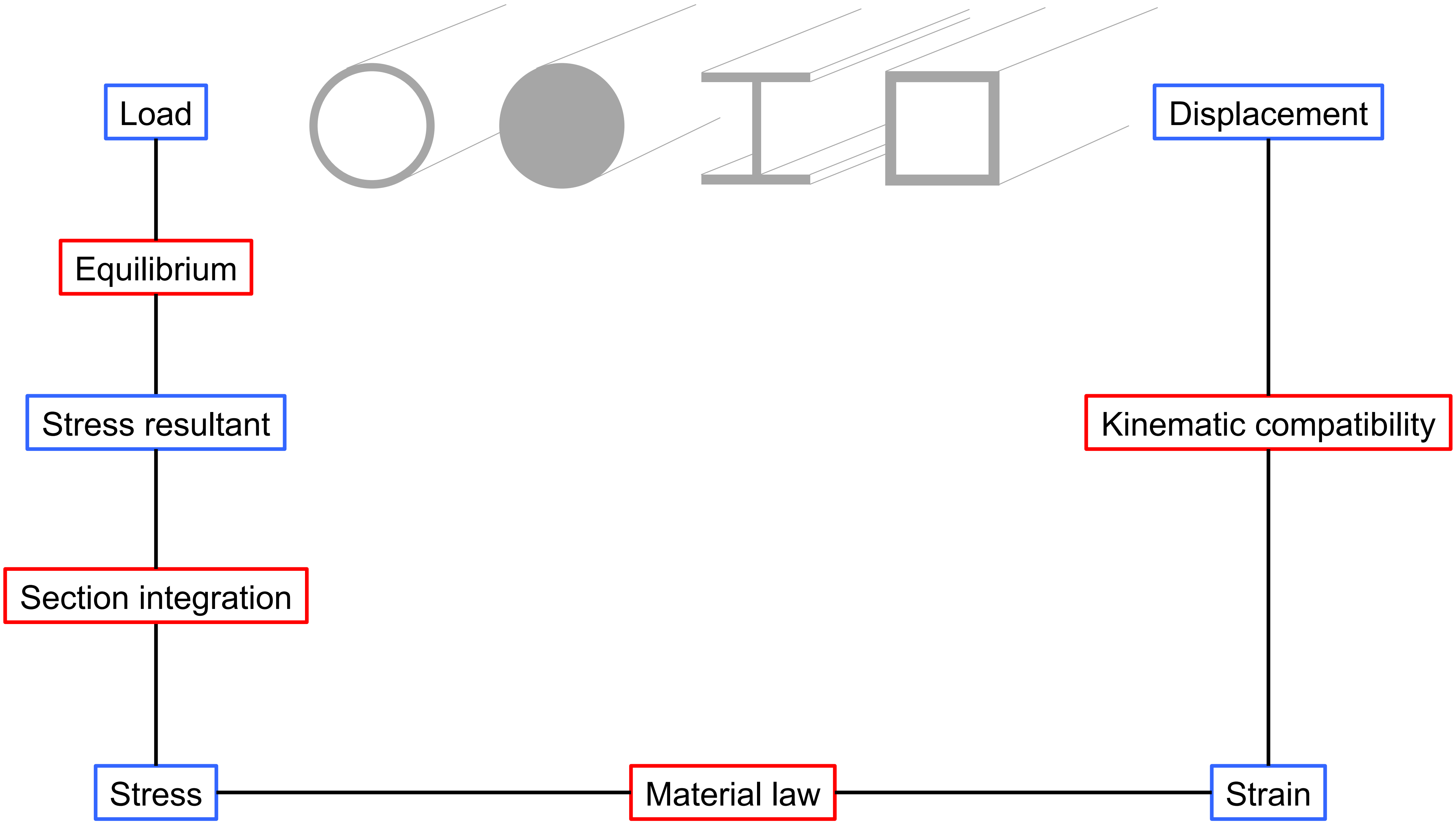
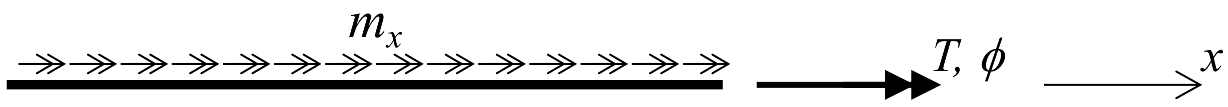
Two Types of Torsion



Ingredients

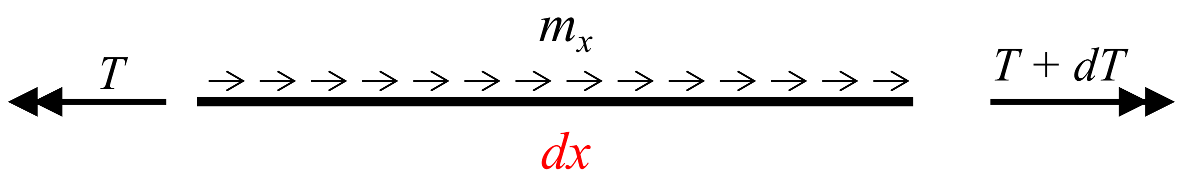


Notation



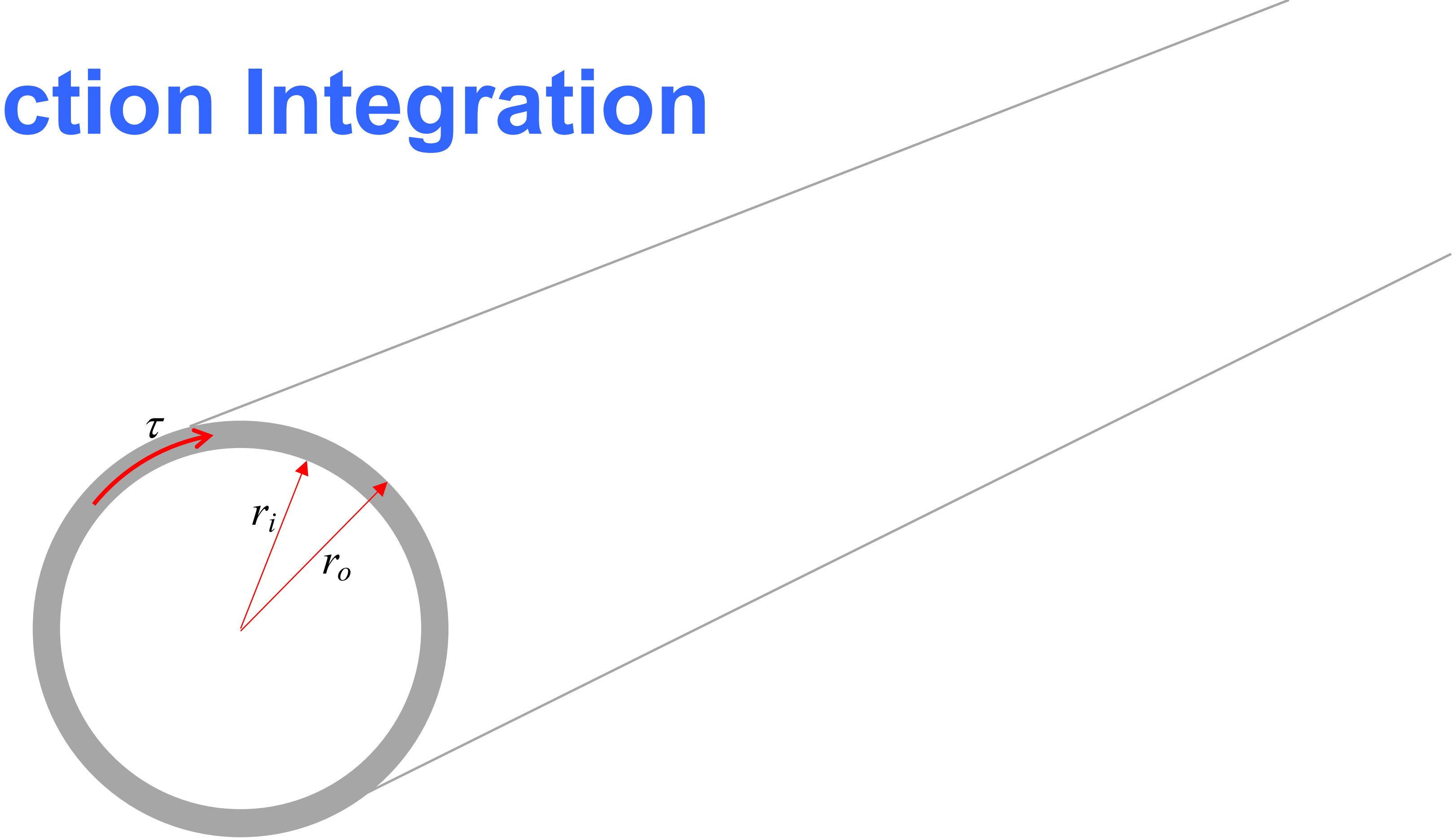
x	=	longitudinal axis
m_x	=	distributed torque along the member
T	=	torque, resultant of shear stress
ϕ	=	rotation about the x-axis
J	=	cross-section constant for St. Venant torsion
G	=	shear modulus = $E/(2(1+\nu))$
τ	=	shear stress
γ	=	shear strain

Equilibrium



$$m_x \cdot dx - T + T + dT = 0 \quad \Rightarrow \quad m_x = -\frac{dT}{dx}$$

Section Integration



$$T = \int_{r_i}^{r_o} (2\pi r) \cdot \tau \cdot r \cdot \mathrm{d}r$$

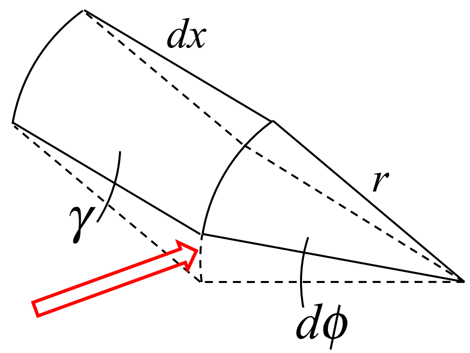
Material Law

$$\tau = G \cdot \gamma$$

$$G = \frac{E}{2(1 + \nu)}$$

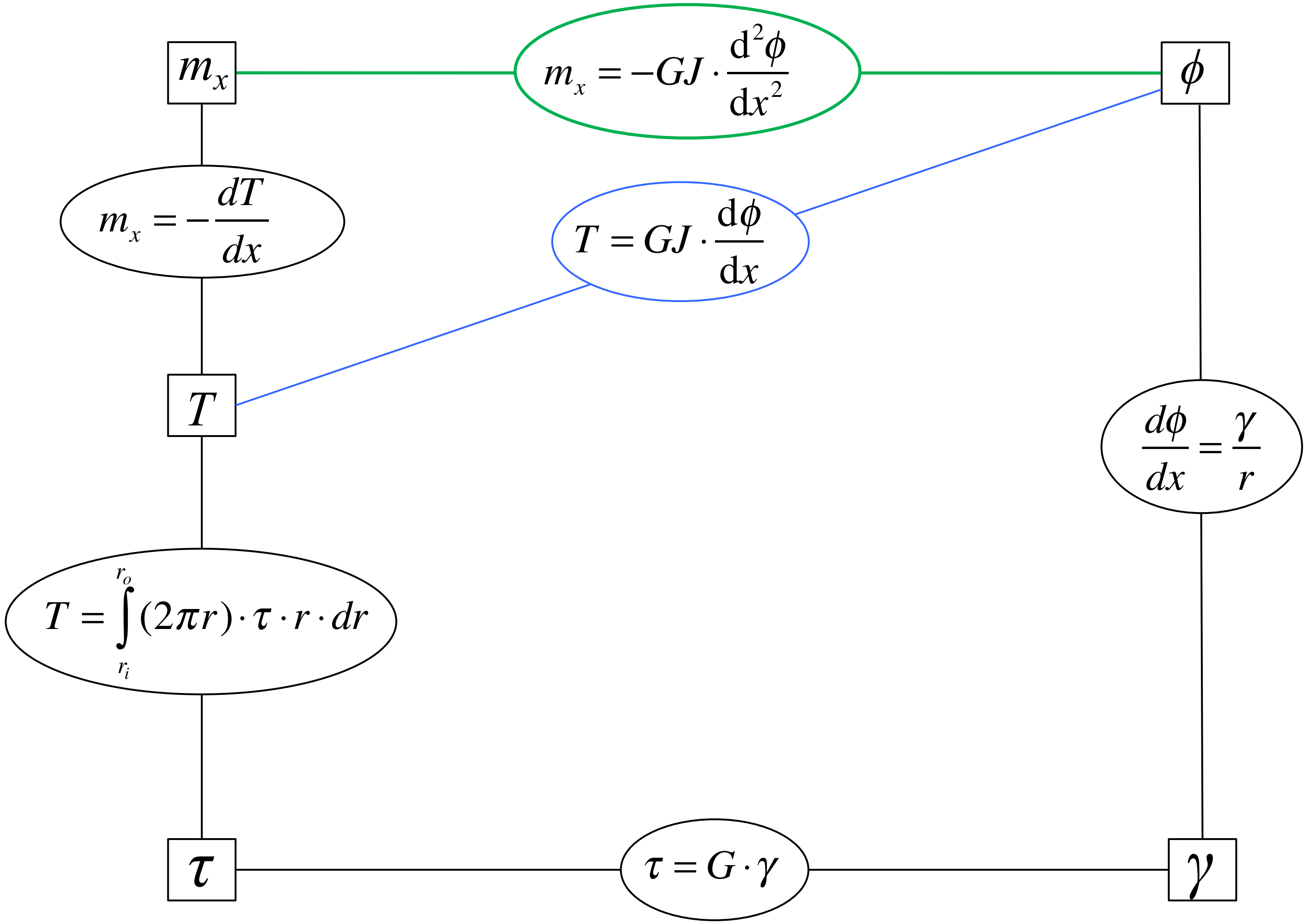
Kinematic Compatibility

$$\frac{\mathrm{d}\phi}{\mathrm{d}x} = \frac{\gamma}{r}$$



Summary

$$J = \int_{r_i}^{r_o} 2\pi r^3 \, \mathrm{d}r = \frac{\pi}{2} \cdot \left(r_o^4 - r_i^4\right)$$



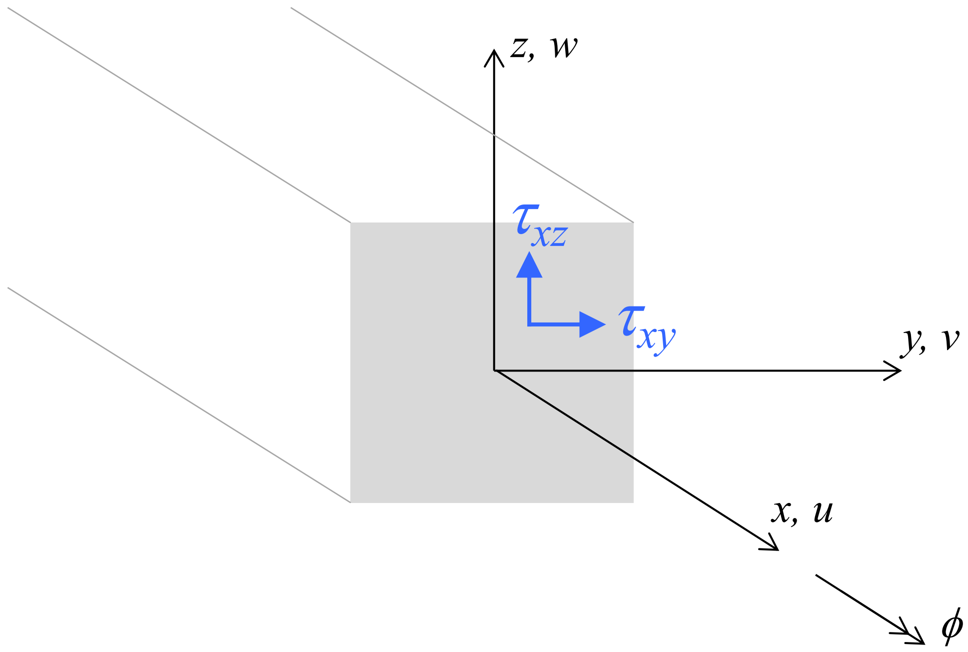
Prandtl's Stress Function

$$P(y,z)$$

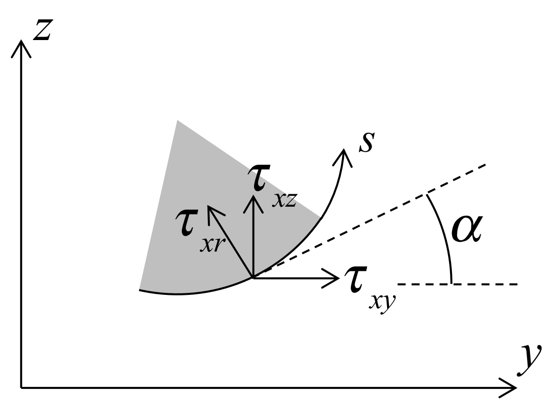
$$\tau_{xy} = \frac{\partial P}{\partial z} = P_{,z}$$
$$\tau_{xz} = -\frac{\partial P}{\partial y} = -P_{,y}$$

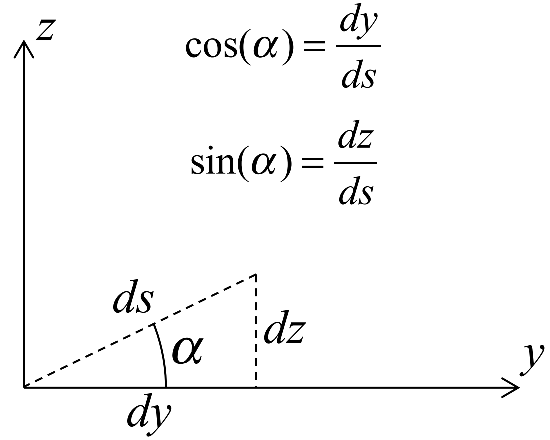
$$\sigma_{xx,x} + \tau_{yx,y} + \tau_{zx,z} = 0$$
$$\tau_{xy,x} + \sigma_{yy,y} + \tau_{zy,z} = 0$$
$$\tau_{xz,x} + \tau_{yz,y} + \sigma_{zz,z} = 0$$

$$\sigma_{xx,x} + \tau_{yx,y} + \tau_{zx,z} = 0 + P_{,zy} - P_{,yz} = 0$$
$$\tau_{xy,x} + \sigma_{yy,y} + \tau_{zy,z} = P_{,zx} + 0 + 0 = 0$$
$$\tau_{xz,x} + \tau_{yz,y} + \sigma_{zz,z} = -P_{,zx} + 0 + 0 = 0$$



Boundary Conditions for Stress Function





$$\cos(\alpha) = \frac{dy}{ds}$$

$$\sin(\alpha) = \frac{dz}{ds}$$

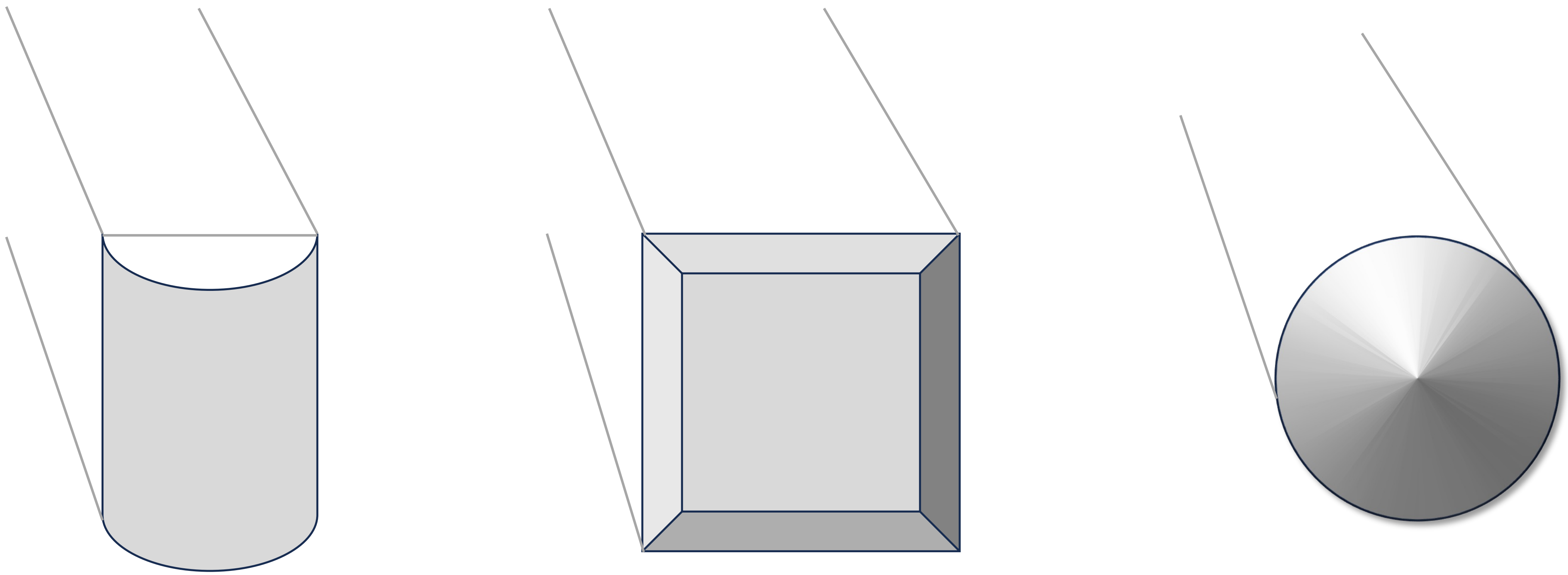
$$\tau_{xr} = 0$$

$$\tau_{xr} = \frac{\partial P(r,s)}{\partial s}$$

$$\frac{\partial P(r,s)}{\partial s} = 0$$

$$\tau_{xr} = \tau_{xz} \cdot \cos(\alpha) - \tau_{xy} \cdot \sin(\alpha) = -\left(\frac{\partial P}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial P}{\partial z}\frac{\partial z}{\partial s} \right) = \frac{\partial P}{\partial s} = 0$$

Membranes



Section Integration

$$T = \int_A \left(\tau_{xz} \cdot y - \tau_{xy} \cdot z \right) \mathrm{d}A$$

$$T = -\int_A \left(\frac{\partial P}{\partial y} \cdot y + \frac{\partial P}{\partial z} \cdot z \right) \mathrm{d}A$$

$$\begin{aligned} T &= -\left(\int_A \frac{\partial P}{\partial y} \cdot y \cdot \mathrm{d}A \right) - \left(\int_A \frac{\partial P}{\partial z} \cdot z \cdot \mathrm{d}A \right) \\ &= -\left(\oint P \cdot y \cdot \mathrm{d}\Gamma - \int_A P \cdot \mathrm{d}A \right) - \left(\oint P \cdot z \cdot \mathrm{d}\Gamma - \int_A P \cdot \mathrm{d}A \right) \\ &= -\underbrace{\oint P \cdot y \cdot \mathrm{d}\Gamma}_{0} - \underbrace{\oint P \cdot z \cdot \mathrm{d}\Gamma}_{0} + \int_A P \cdot \mathrm{d}A + \int_A P \cdot \mathrm{d}A \\ &= 2 \cdot \int_A P \cdot \mathrm{d}A \end{aligned}$$

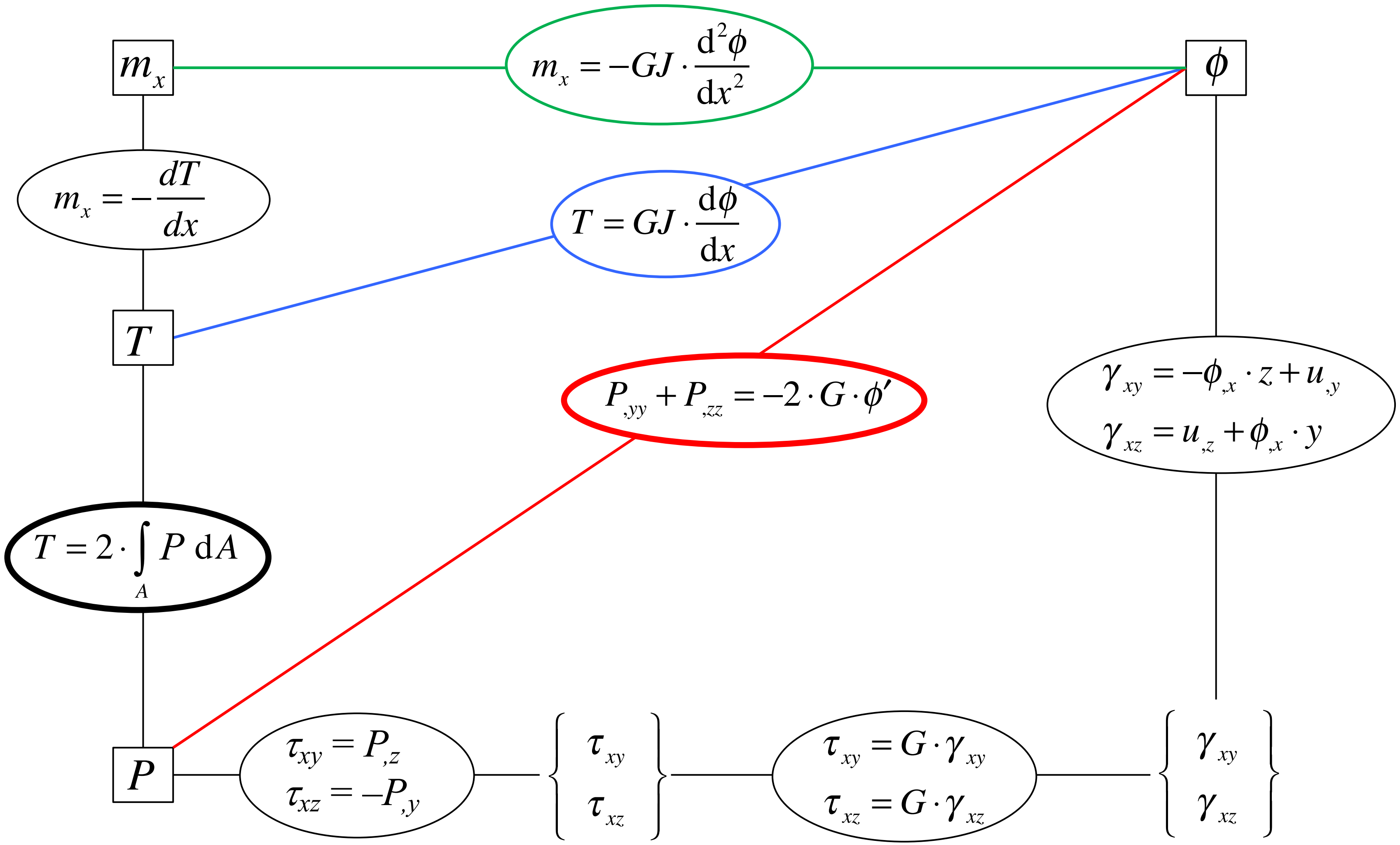
Kinematic Compatibility

$$v = -\phi \cdot z$$
$$w = \phi \cdot y$$

$$\varepsilon_x = \frac{du}{dx} = 0$$
$$\varepsilon_y = \frac{dv}{dy} = 0$$
$$\varepsilon_z = \frac{dw}{dz} = 0$$

$$\gamma_{xy} = \frac{dv}{dx} + \frac{du}{dy} = -\frac{d\phi}{dx} \cdot z + \frac{du}{dy}$$
$$\gamma_{xz} = \frac{du}{dz} + \frac{dw}{dx} = \frac{du}{dz} + \frac{d\phi}{dx} \cdot y$$
$$\gamma_{yz} = \frac{dw}{dy} + \frac{dv}{dz} = 0$$

Summary



$$P_{,z} = \tau_{xy} = G \cdot \gamma_{xy} = G \cdot \left(-\phi_{,x} \cdot z + u_{,y} \right)$$
$$P_{,y} = -\tau_{xz} = -G \cdot \gamma_{xz} = -G \cdot \left(u_{,z} + \phi_{,x} \cdot y \right)$$

$\Rightarrow$

$$\frac{\partial^2 P(y,z)}{\partial y^2} + \frac{\partial^2 P(y,z)}{\partial z^2} \equiv P_{,yy} + P_{,zz} \equiv \nabla^2 P(y,z) = -2 \cdot G \cdot \phi'$$

Cross-section Analysis

Cross-section constant, J

Shear stress τ_{xy} and τ_{xz}

$$T = GJ \cdot \frac{\mathrm{d}\phi}{\mathrm{d}x}$$

$$\underbrace{\left(2 \cdot \int_A P \cdot \mathrm{d}A\right)}_{T} = GJ \cdot \underbrace{\left(-\frac{P_{,yy} + P_{,zz}}{2 \cdot G}\right)}_{\phi_{,x}}$$

$$T = 2 \cdot \int_A P \, \mathrm{d}A \qquad P_{,yy} + P_{,zz} = -2 \cdot G \cdot \phi'$$

$$J = -\frac{4 \cdot V}{\nabla^2 P}$$

More lectures:

Terje’s Toobox:

terje.civil.ubc.ca