A short course on

The Finite Element Method

This video:

Four-node Quadrilateral Elements (Quad4)

Terje's Toolbox is freely available at terje.civil.ubc.ca
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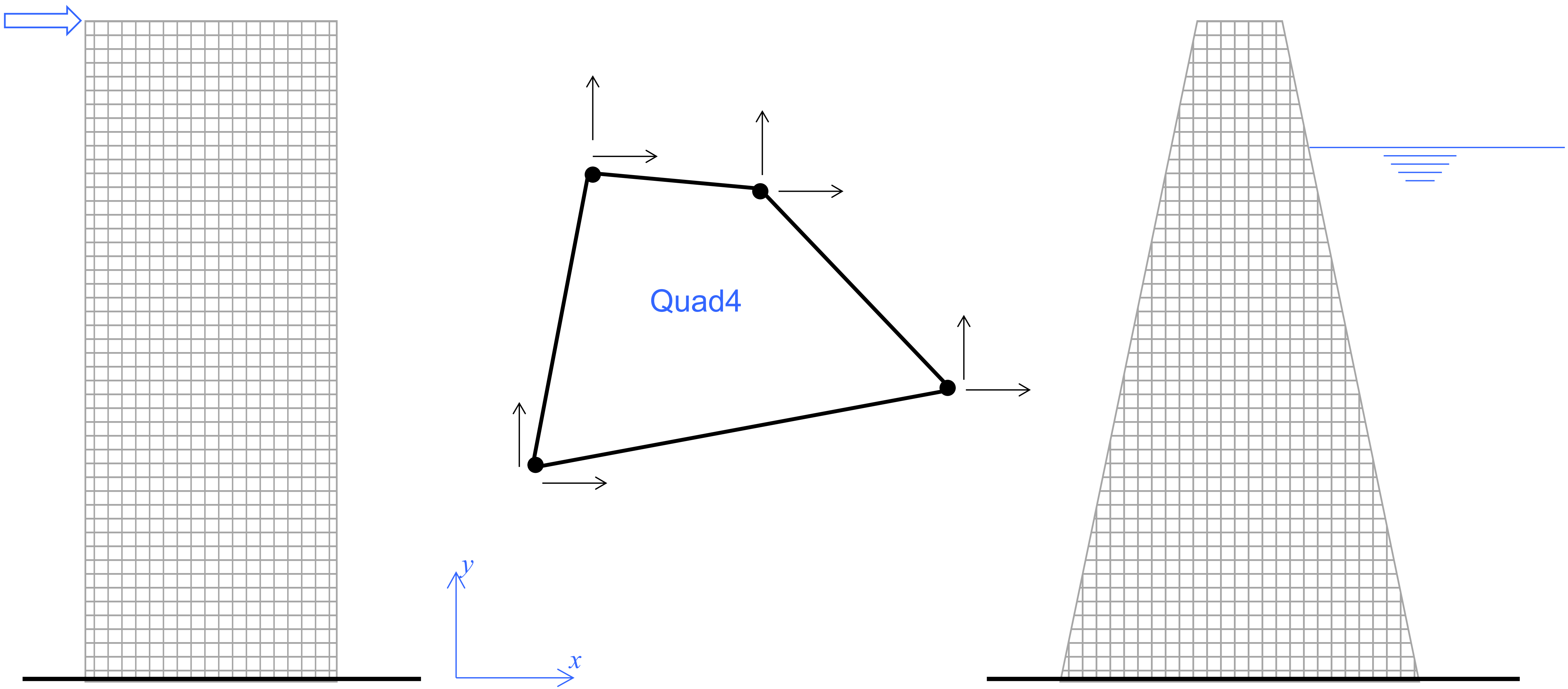
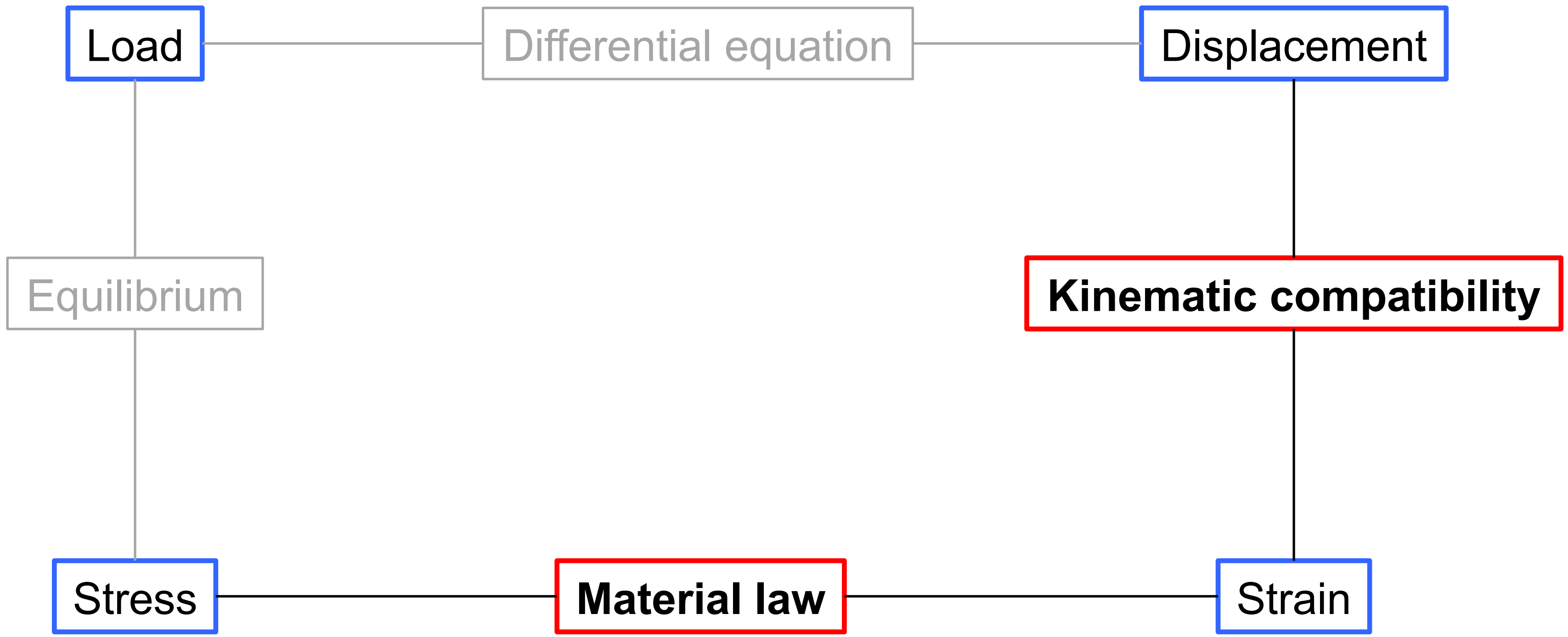
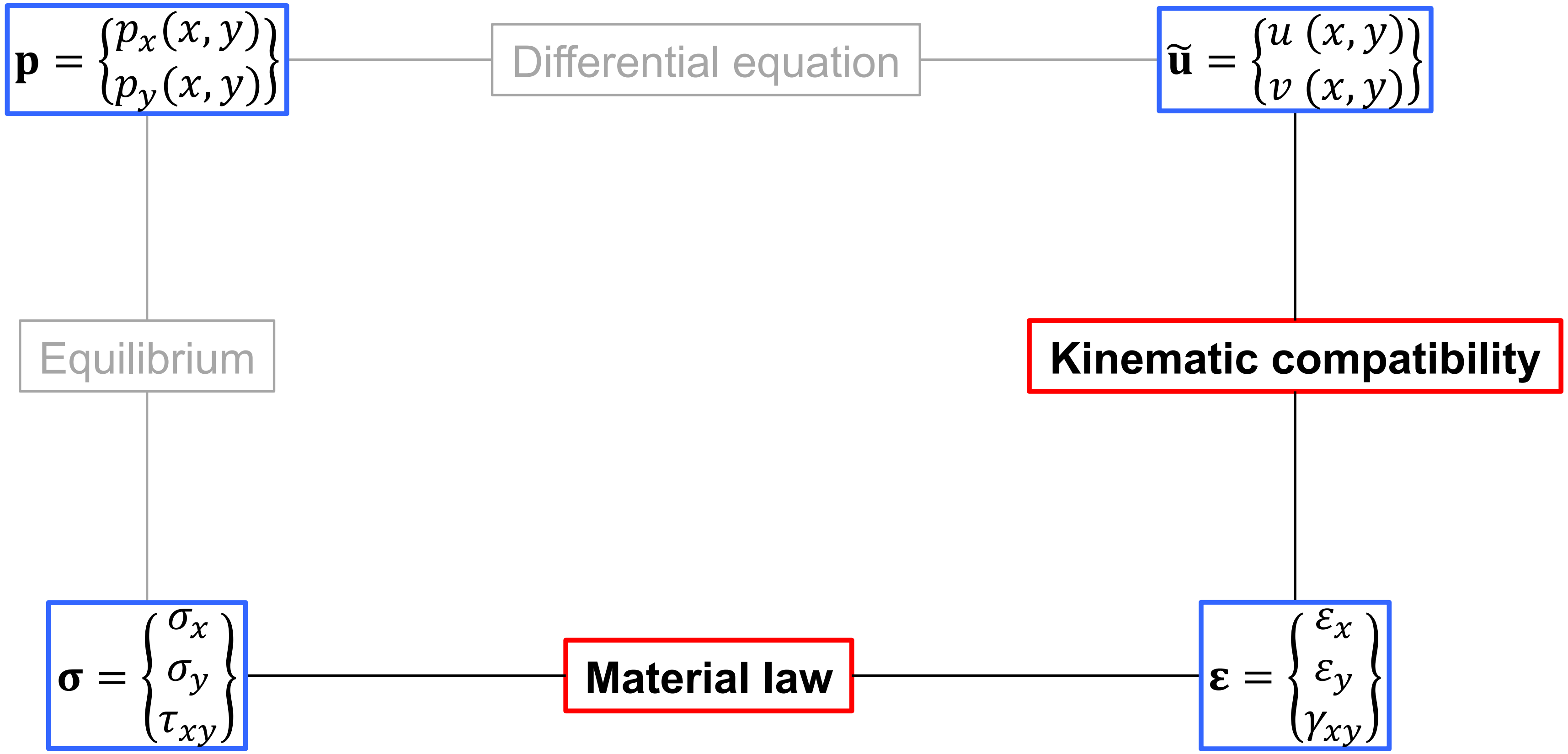
2D Continuum Elements



Boundary Value Problem



Boundary Value Problem



Finite Element Procedure

1. Start with the weak form of the boundary value problem = principle of virtual displacements
2. Substitute material law
3. Substitute kinematic compatibility
4. Substitute the shape function discretization of the displacements
5. Integrate to obtain **K** and **F**

Virtual Work + Material Law

$$\int_V \delta \boldsymbol{\varepsilon}^T \boldsymbol{\sigma}\, dV - \int_V \delta \tilde{\mathbf{u}}^T \mathbf{p}\, dV = 0$$

$$\boldsymbol{\sigma} = \mathbf{D}\boldsymbol{\varepsilon}$$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{1-\nu^2} \cdot \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}$$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \cdot \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}$$

$$\int_V \delta \boldsymbol{\varepsilon}^T \mathbf{D} \boldsymbol{\varepsilon}\, dV - \int_V \delta \tilde{\mathbf{u}}^T \mathbf{p}\, dV = 0$$

+ Kinematic Compatibility + Discretize

$$\boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} \cdot \begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix} = \nabla \tilde{\mathbf{u}}$$

$$\tilde{\mathbf{u}} = \mathbf{N}\mathbf{u}$$

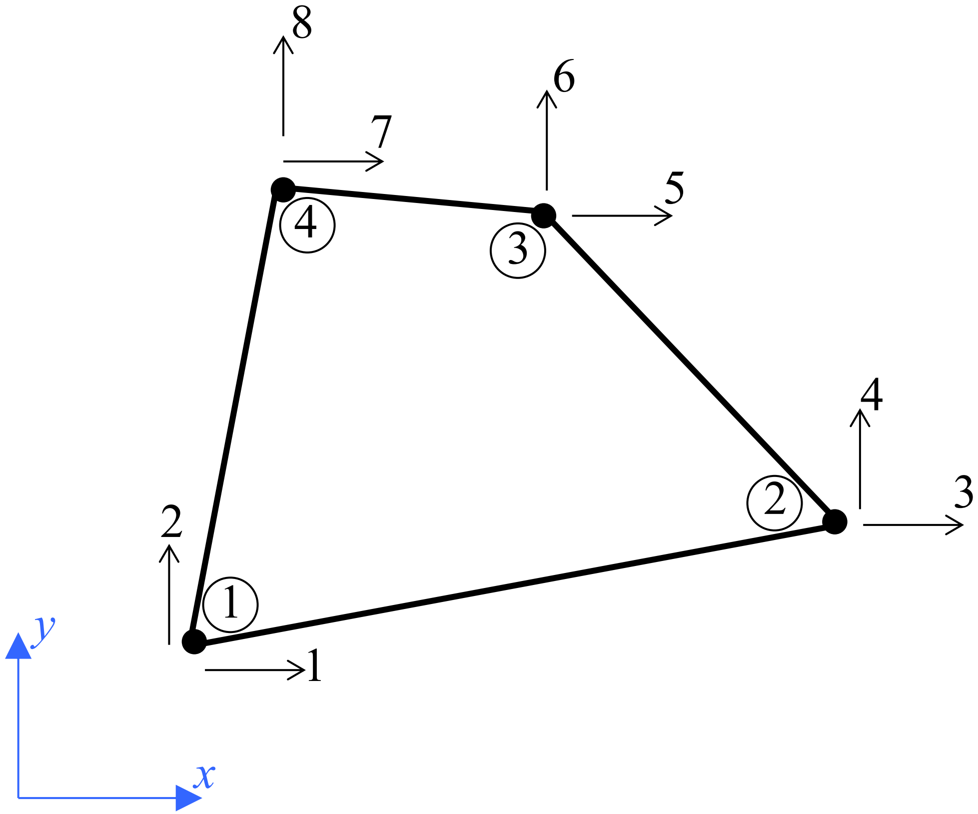
$$\varepsilon = \nabla \tilde{\mathbf{u}} = \nabla \mathbf{N}\mathbf{u} \equiv \mathbf{B}\mathbf{u} \quad \Leftarrow \quad \mathbf{B} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} \left[\begin{array}{cc|cc|cc|cc} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{array} \right]$$

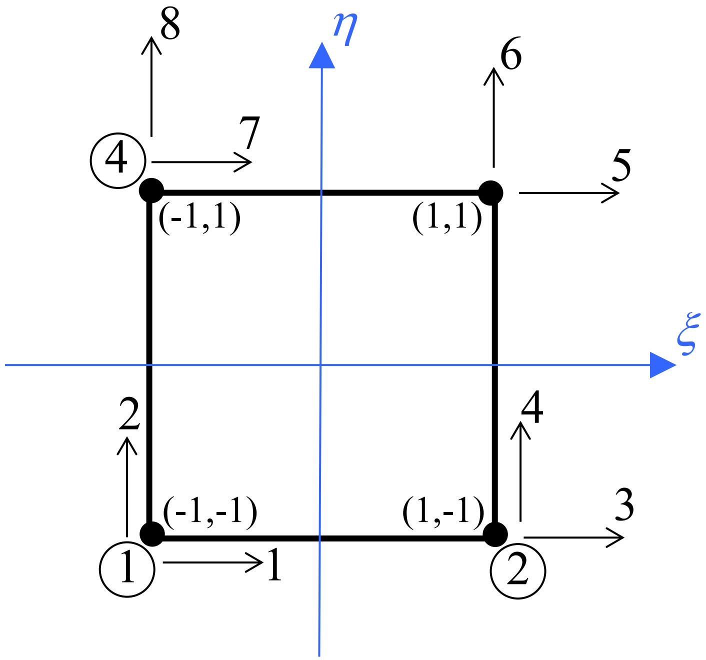
$$\int_V (\mathbf{B}\delta\mathbf{u})^T \mathbf{D}(\mathbf{B}\mathbf{u})\, dV - \int_V (\mathbf{N}\delta\mathbf{u})^T \mathbf{p}\, dV = 0$$

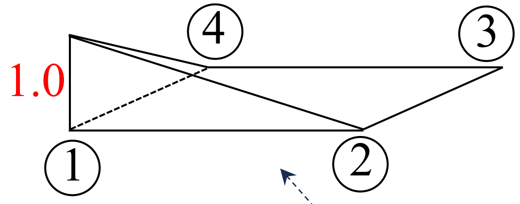
$$\delta\mathbf{u}^T \left(\left(\int_V \mathbf{B}^T \mathbf{D}\mathbf{B}\, dV \right) \mathbf{u} - \int_V \mathbf{N}^T \mathbf{p}\, dV \right) = 0$$

$$\underbrace{\left(\int_V \mathbf{B}^T \mathbf{D}\mathbf{B}\, dV \right)}_{\mathbf{K}} \mathbf{u} = \underbrace{\int_V \mathbf{N}^T \mathbf{p}\, dV}_{\mathbf{F}}$$

Shape Functions







$$\begin{Bmatrix} \tilde{u}(\xi,\eta) \\ \tilde{v}(\xi,\eta) \end{Bmatrix} = \left[\begin{array}{cc|cc|cc|cc} N_1(\xi,\eta) & 0 & N_2(\xi,\eta) & 0 & N_3(\xi,\eta) & 0 & N_4(\xi,\eta) & 0 \\ 0 & N_1(\xi,\eta) & 0 & N_2(\xi,\eta) & 0 & N_3(\xi,\eta) & 0 & N_4(\xi,\eta) \end{array}\right] \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \end{Bmatrix}$$

$$N_1(\xi,\eta) = \frac{1}{4}(1-\xi)(1-\eta)$$

$$N_2(\xi,\eta) = \frac{1}{4}(1+\xi)(1-\eta)$$

$$N_3(\xi,\eta) = \frac{1}{4}(1+\xi)(1+\eta)$$

$$N_4(\xi,\eta) = \frac{1}{4}(1-\xi)(1+\eta)$$

Integral Transformation

$$\int_x f(x)\,dx = \int_\xi f(x(\xi))\frac{dx}{d\xi}\,d\xi$$

$$\int_y \int_x f(x,y)\,dx\,dy = \int_\eta \int_\xi f\big(x(\xi,\eta),y(\xi,\eta)\big)\left|\mathbf{J}\right|d\xi\,d\eta$$

$$\mathbf{J} = \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial x}{\partial \eta} \\ \dfrac{\partial y}{\partial \xi} & \dfrac{\partial y}{\partial \eta} \end{bmatrix}$$

$$\mathbf{K} = h \int_{-1}^{1} \int_{-1}^{1} \mathbf{B}^{\mathrm{T}}\mathbf{D}\mathbf{B}\left|\mathbf{J}\right| d\xi\;\, d\eta$$

Chain Rule

$$\frac{\partial N_i\left(\xi(x,y),\eta(x,y)\right)}{\partial x} = \frac{\partial N_i}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial N_i}{\partial \eta}\frac{\partial \eta}{\partial x}$$

$$\frac{\partial N_i\left(\xi(x,y),\eta(x,y)\right)}{\partial y} = \frac{\partial N_i}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial N_i}{\partial \eta}\frac{\partial \eta}{\partial y}$$

$$\mathbf{J}^{-1} = \begin{bmatrix} \dfrac{\partial \xi}{\partial x} & \dfrac{\partial \xi}{\partial y} \\ \dfrac{\partial \eta}{\partial x} & \dfrac{\partial \eta}{\partial y} \end{bmatrix}$$

Iso-parametric Formulation

$$x = N_1 x_1 + N_2 x_2 + N_3 x_3 + N_4 x_4$$

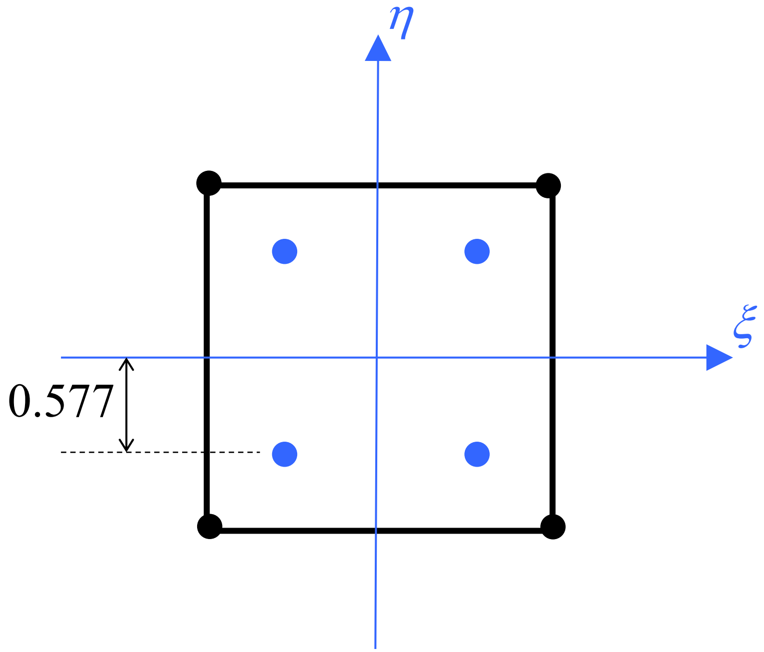
$$y = N_1 y_1 + N_2 y_2 + N_3 y_3 + N_4 y_4$$

$$\frac{\partial x}{\partial \xi} = \sum \frac{\partial N_i}{\partial \xi} x_i \qquad \frac{\partial x}{\partial \eta} = \sum \frac{\partial N_i}{\partial \eta} x_i$$

$$\frac{\partial y}{\partial \xi} = \sum \frac{\partial N_i}{\partial \xi} y_i \qquad \frac{\partial y}{\partial \eta} = \sum \frac{\partial N_i}{\partial \eta} y_i$$

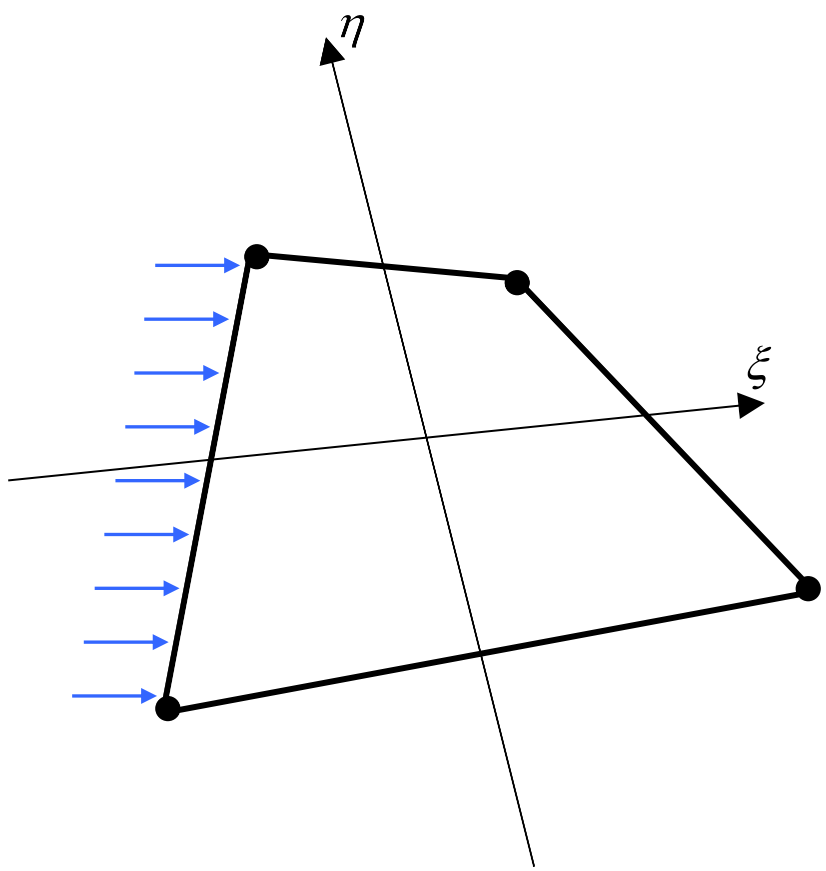
Quadrature

$$\int_{-1}^{1} f(\xi)\ d\xi = \sum_{i=1}^{N} w_i \cdot f(\xi_i)$$



$$\mathbf{K} = h\int_{-1}^{1}\int_{-1}^{1} \mathbf{B}^T\mathbf{D}\ \mathbf{B}|\mathbf{J}|\,\mathrm{d}\xi\, d\eta = h\sum_{i=1}^{N}\sum_{j=1}^{N} w_i w_j \left(\mathbf{B}^T\mathbf{D}\ \mathbf{B}\ |\mathbf{J}|\right)_{ij}$$

Consistent Load Vector



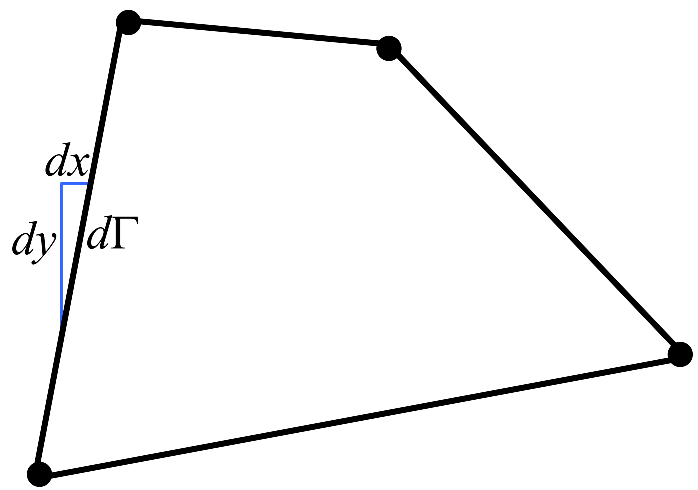
$$\mathbf{F} = \int_V \mathbf{N}^T \mathbf{p}\, dV = \int_\Gamma \mathbf{N}^T \mathbf{p}\, d\Gamma = \int_{-1}^{1} \mathbf{N}^T \mathbf{p} \frac{d\Gamma}{d\eta} d\eta$$

$$d\Gamma^2 = dx^2 + dy^2$$

$$dx = \frac{dx}{d\eta} d\eta$$

$$dy = \frac{dy}{d\eta} d\eta$$

$$d\Gamma = d\eta \cdot \sqrt{\left(\frac{dx}{d\eta}\right)^2 + \left(\frac{dy}{d\eta}\right)^2}$$



More lectures:

Terje’s Toobox:

terje.civil.ubc.ca