A short course on

Probabilities and Random Variables

This video:

Multiple Random Variables: Joint Distributions & Matrix Notation

Terje's Toolbox is freely available at terje.civil.ubc.ca
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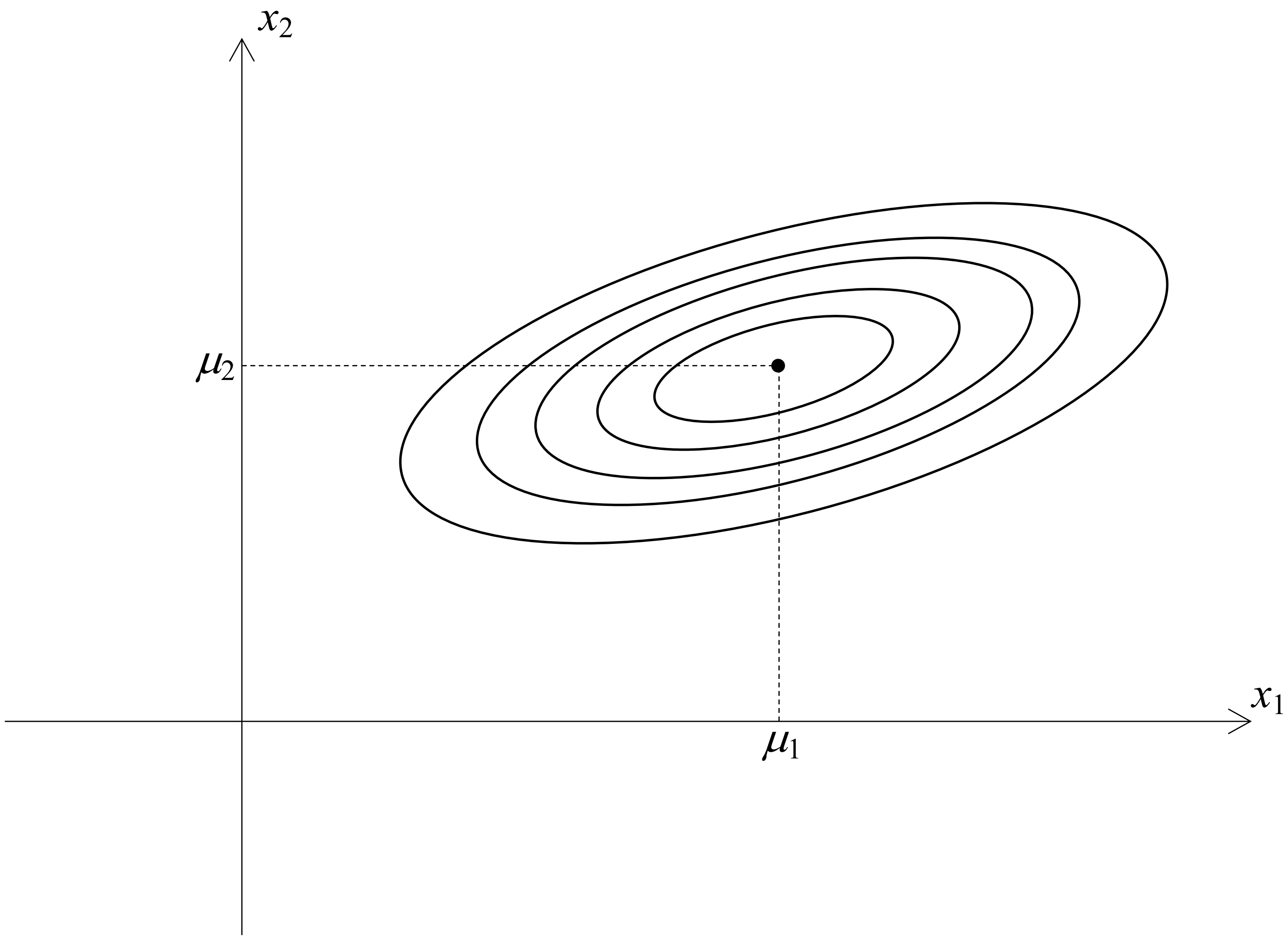
Joint Distribution



Properties

$$f(x,y)\cdot dx\cdot dy = \mathrm{P}\left(x < X \le x+dx \cap y < Y \le y+dy\right)$$

$$f(x,y) \ge 0$$

$$f(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$$

$$f(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$$

Marginal distributions from joint distribution

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$$

$$F(x,y) = P(X \le x \cap Y \le y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f(x,y)\,dx\,dy$$

$$f(x,y) = \frac{\partial^2 F(x,y)}{\partial x \partial y}$$

Moments

Mean product:

$$E[XY] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x \cdot y \cdot f(x,y)\,dx\,dy$$

Covariance:

$$\mathrm{Cov}[X,Y] = E[(X-\mu_X)(Y-\mu_Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x-\mu_X)\cdot(y-\mu_Y)\cdot f(x,y)\,dx\,dy$$

Poem

Covariance is mean product minus the product of the means

$$\mathrm{Cov}[X,Y] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(x-\mu_X)\cdot(y-\mu_Y)\cdot f(x,y)\,dx\,dy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(x\cdot y+\mu_X\cdot\mu_Y - x\cdot\mu_Y - y\cdot\mu_X)\cdot f(x,y)dx\,dy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}x\cdot y\cdot f(x,y)dx\,dy + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mu_X\cdot\mu_Y\cdot f(x,y)dx\,dy - \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}x\cdot\mu_Y\cdot f(x,y)dx\,dy - \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}y\cdot\mu_X\cdot f(x,y)dx\,dy$$

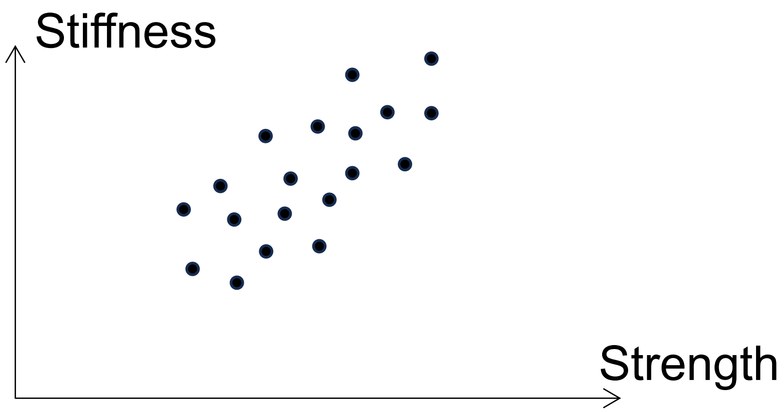
$$= \mathrm{E}[XY] + \mu_X\cdot\mu_Y\cdot\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(x,y)\,dx\,dy - \mu_Y\cdot\int_{-\infty}^{\infty}x\cdot\int_{-\infty}^{\infty}f(x,y)\,dx\,dy - \mu_X\cdot\int_{-\infty}^{\infty}y\cdot\int_{-\infty}^{\infty}f(x,y)\,dx\,dy$$

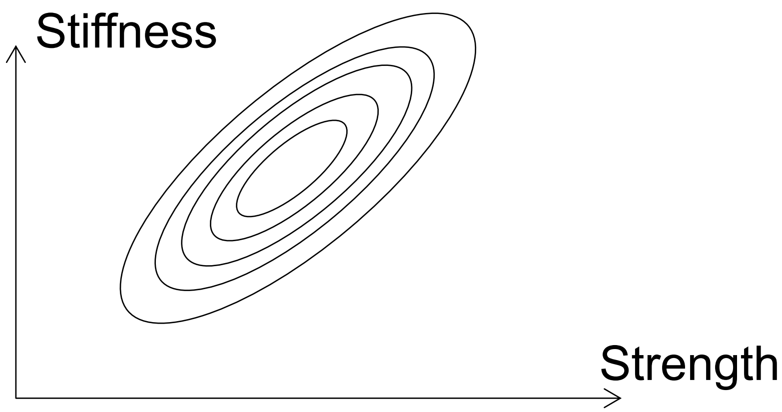
$$= \mathrm{E}[XY] + \mu_X\cdot\mu_Y\cdot 1 - \mu_Y\cdot\int_{-\infty}^{\infty}x\cdot f(x)dx - \mu_X\cdot\int_{-\infty}^{\infty}y\cdot f(y)\,dy$$

$$= \mathrm{E}[XY] + \mu_X\cdot\mu_Y\cdot 1 - \mu_Y\cdot\mu_X - \mu_X\cdot\mu_Y = \mathrm{E}[XY] - \mu_X\cdot\mu_Y$$

Correlation Coefficient

$$\rho_{XY} = \frac{\mathrm{Cov}[X,Y]}{\sigma_X \sigma_Y} = \frac{\mathrm{E}[(X-\mu_X)(Y-\mu_Y)]}{\sigma_X \sigma_Y} = \mathrm{E}\left[\frac{X-\mu_X}{\sigma_X} \cdot \frac{Y-\mu_Y}{\sigma_Y}\right]$$





Vector-Matrix Notation

$$\mathbf{M_X} = \begin{Bmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_n \end{Bmatrix}$$

$$\mathbf{\Sigma_{XX}} = \begin{bmatrix} \sigma_1^2 & \rho_{12}\cdot\sigma_1\cdot\sigma_2 & \rho_{13}\cdot\sigma_1\cdot\sigma_3 \\ \rho_{12}\cdot\sigma_1\cdot\sigma_2 & \sigma_2^2 & \rho_{23}\cdot\sigma_2\cdot\sigma_3 \\ \rho_{13}\cdot\sigma_1\cdot\sigma_3 & \rho_{23}\cdot\sigma_2\cdot\sigma_3 & \sigma_3^2 \end{bmatrix}$$

Later: $f(\mathbf{x}),\ h(\mathbf{x}),\ g(\mathbf{x}),\ \nabla g(\mathbf{x})$

$\mathbf{\Sigma_{XX}} = \mathbf{D_{XX}}\mathbf{R_{XX}}\mathbf{D_{XX}}$, where **D** is diagonal matrix with standard deviations

$$\mathbf{R_{XX}} = \begin{bmatrix} 1 & \rho_{12} & \rho_{13} \\ \rho_{12} & 1 & \rho_{23} \\ \rho_{13} & \rho_{23} & 1 \end{bmatrix}$$

Joint Normal

General:

$$f(\mathbf{x}) = \frac{1}{\sqrt{(2\pi)^n} \cdot \sqrt{\det(\mathbf{\Sigma}_{XX})}} \cdot \exp\left(\frac{1}{2}(\mathbf{x} - \mathbf{M}_X)^T \mathbf{\Sigma}_{XX}^{-1} (\mathbf{x} - \mathbf{M}_X)\right)$$

Bi-variate:

$$f(x_1, x_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \cdot \exp\left(-\frac{z}{2(1-\rho)^2}\right)$$

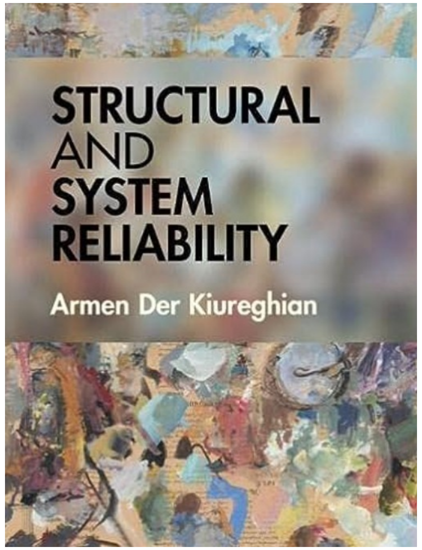
$$z = \left(\frac{x_1 - \mu_1}{\sigma_1}\right)^2 + \left(\frac{x_2 - \mu_2}{\sigma_2}\right)^2 - \frac{2\rho(x_1 - \mu_1)(x_2 - \mu_2)}{\sigma_1\sigma_2}$$

Joint standard normal:

$$\varphi(\mathbf{y}) = \frac{1}{\sqrt{(2\pi)^n}} \cdot \exp\left(-\frac{1}{2}\mathbf{y}^T \mathbf{y}\right)$$

Joint Lognormal

Section 3.3



Joint from Conditional

$$f(x \mid y) \cdot dx = \frac{f(x,y) \cdot dx \cdot dy}{f(y) \cdot dy} \Rightarrow f(x \mid y) = \frac{f(x,y)}{f(y)}$$

$$f(x,y) = f(x \mid y) \cdot f(y) = f(y \mid x) \cdot f(x)$$

What is statistical independence for random variables?

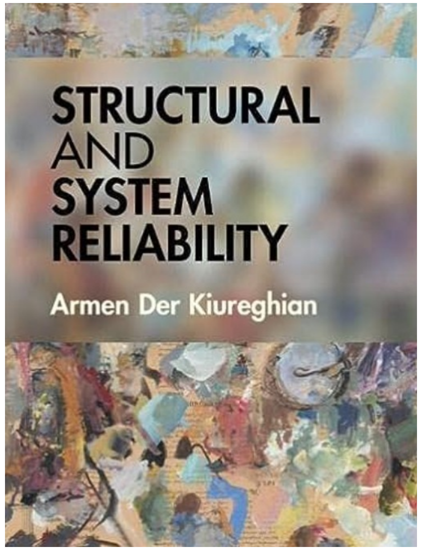
Consequence of statistical independence: $f(x,y) = f(x)\, f(y)$

What is the difference between correlation and statistical dependence?

Joint from Marginals + Correlation

Morgenstern

Nataf



More in the short course on Structural Reliability

More lectures:

Terje’s Toobox:

terje.civil.ubc.ca