A short course on

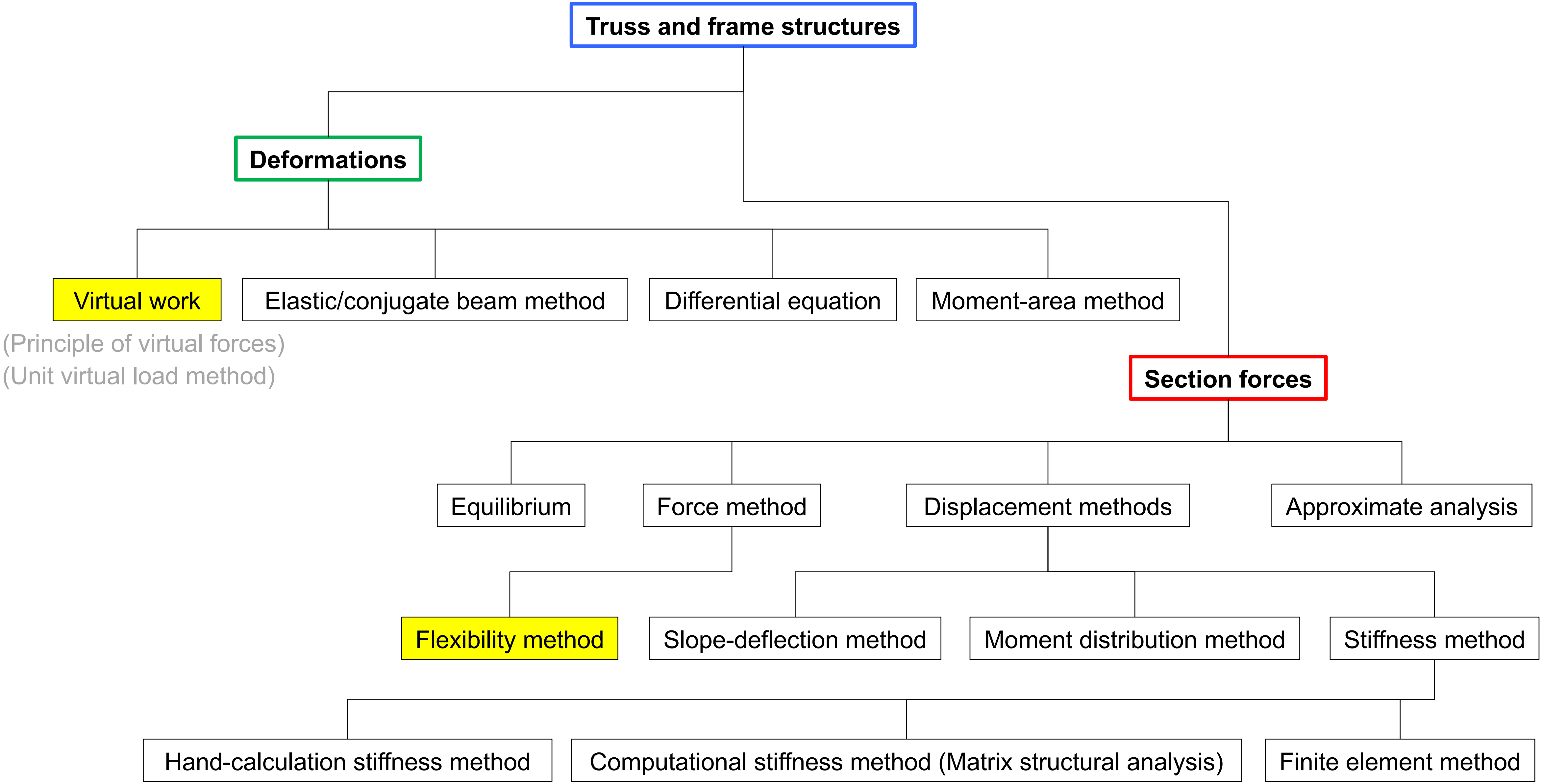
Indeterminate Structures

This video:

Virtual Work

Terje's Toolbox is freely available at terje.civil.ubc.ca
It is created and maintained by Professor Terje Haukaas, Ph.D., P.Eng.,
Department of Civil Engineering, The University of British Columbia (UBC), Vancouver, Canada

Overview of Methods



Why Deformations?

Serviceability limit-states

$\Delta < L/300$

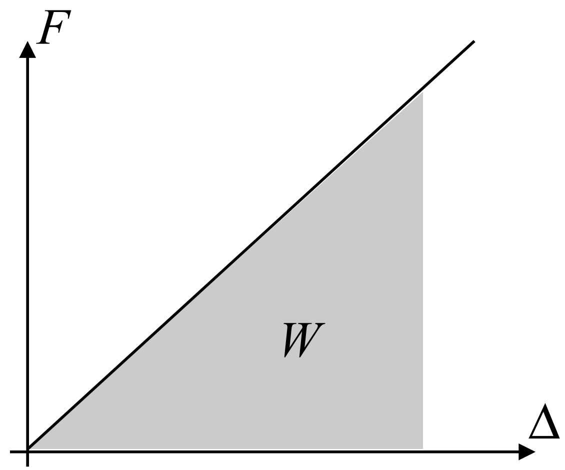
Analysis of indeterminate structures with the flexibility method

Deformations influence internal forces

Real Work

$$W = F \cdot \Delta$$

$$W = M \cdot \theta$$



$$W = \int_0^\Delta F \ \mathrm{d}\Delta$$

$$W = \int_0^\kappa M \ \mathrm{d}\theta$$

$$W = \frac{1}{2} \cdot F \cdot \Delta$$

$$W = \frac{1}{2} \cdot M \cdot \theta$$

Internal Work

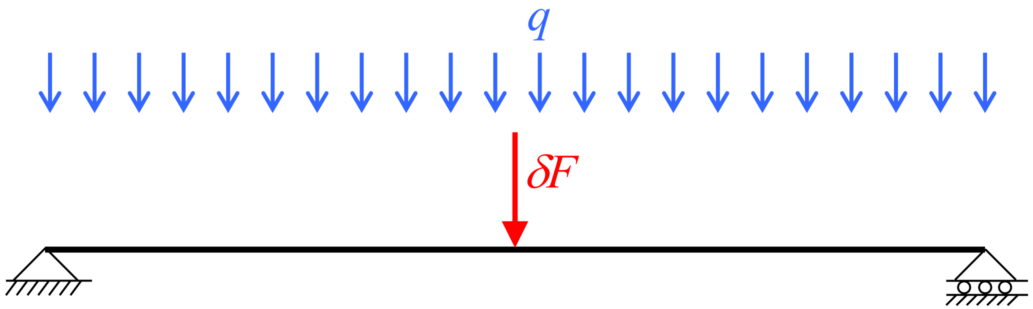
$$U = \int_V \frac{1}{2} \cdot \sigma \cdot \varepsilon \cdot \mathrm{d}V = \int_V \frac{1}{2} \cdot \left(\frac{N}{A}\right) \cdot \left(\frac{\sigma}{E}\right) \cdot \mathrm{d}V = \int_V \frac{1}{2} \cdot \left(\frac{N}{A}\right) \cdot \left(\frac{N}{EA}\right) \cdot \mathrm{d}V$$

$$= \int_0^L \frac{1}{2} \cdot N \cdot \left(\frac{N}{EA}\right) \cdot \mathrm{d}x = \frac{1}{2} \cdot \underbrace{N}_{\text{Force}} \cdot \underbrace{\left(\frac{N}{EA}\right) \cdot L}_{\text{Elongation}}$$

$$U = \int_V \frac{1}{2} \cdot \sigma \cdot \varepsilon \, \mathrm{d}V = \int_V \frac{1}{2} \cdot \left(\frac{M}{I} \cdot z\right) \cdot \left(\frac{\sigma}{E}\right) \mathrm{d}V$$

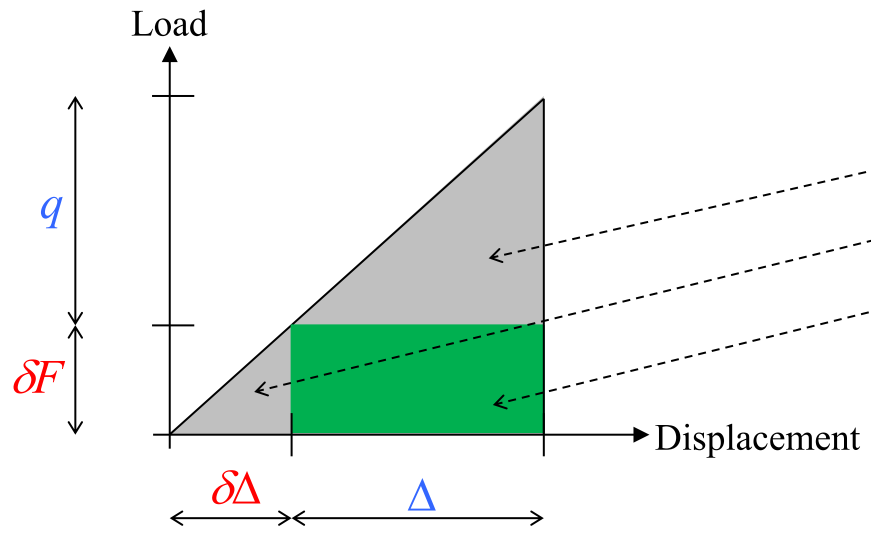
$$= \int_V \frac{1}{2} \cdot \left(\frac{M}{I} \cdot z\right) \cdot \left(\frac{M}{EI} \cdot z\right) \mathrm{d}V = \int_0^L \frac{1}{2} \cdot \underbrace{M}_{\text{Moment}} \cdot \underbrace{\left(\frac{M}{EI}\right)}_{\text{Curvature}} dx$$

Experiment



Virtual Work

External:



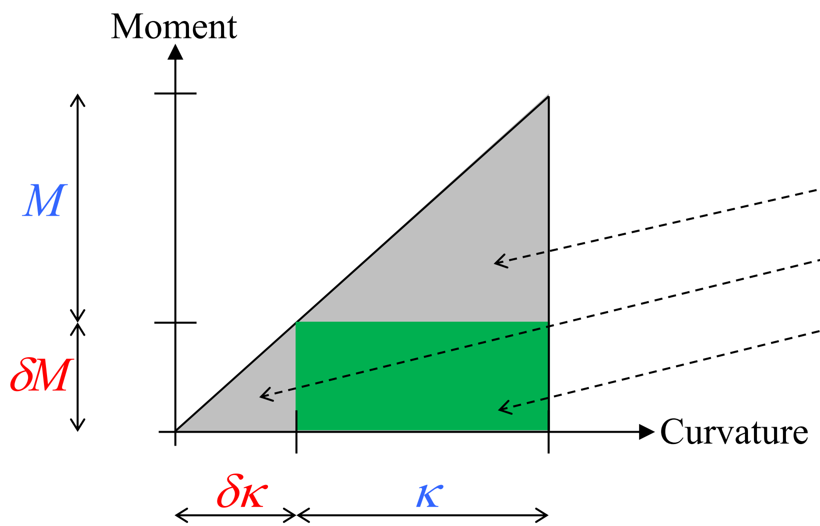
Real work due to application of real loads: $W = \tfrac{1}{2} \cdot q \cdot \Delta$

Virtual work due to application of virtual load: $W = \tfrac{1}{2} \cdot \delta F \cdot \delta\Delta$

Virtual work due to application of real loads: $W = \delta F \cdot \Delta$

$$\delta F \cdot \Delta = \int_0^L \delta M \cdot \kappa \, dx = \int_0^L \delta M \cdot \frac{M}{EI} \, dx$$

Internal:



Real work due to application of real loads: $U = \tfrac{1}{2} \cdot M \cdot \kappa$

Virtual work due to application of virtual load: $U = \tfrac{1}{2} \cdot \delta M \cdot \delta\kappa$

Virtual work due to application of real loads: $U = \delta M \cdot \kappa$

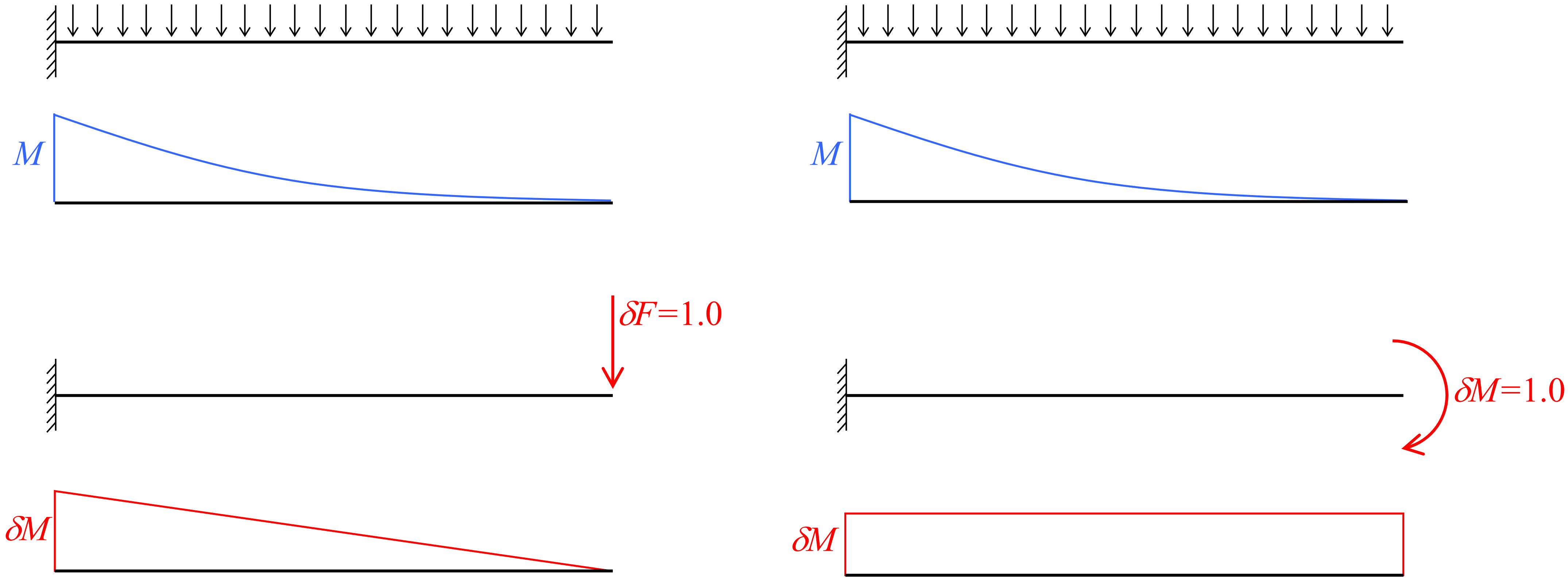
Result

$$\delta F \cdot \Delta = \sum_{\substack{\text{Sum over}\\\text{all members}}} \left(\frac{\delta N \cdot N \cdot L}{EA} + \int_0^L \frac{\delta M \cdot M}{EI} \mathrm{d}x + \int_0^L \frac{\delta V \cdot V}{G \cdot A_v} \mathrm{d}x \right)$$

1.0 * Real displacement = Σ Virtual internal forces * Real internal deformations

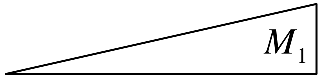
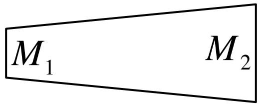
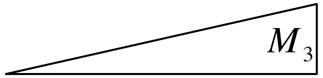
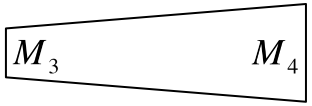
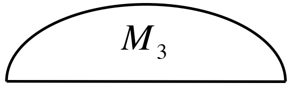
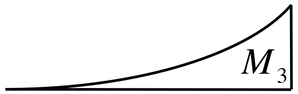
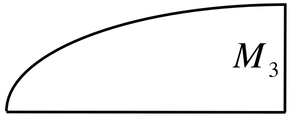
$$\delta M \cdot \theta = \sum_{\substack{\text{Sum over}\\\text{all members}}} \left(\frac{\delta N \cdot N \cdot L}{EA} + \int_0^L \frac{\delta M \cdot M}{EI} \mathrm{d}x + \int_0^L \frac{\delta V \cdot V}{G \cdot A_v} \mathrm{d}x \right)$$

Procedure

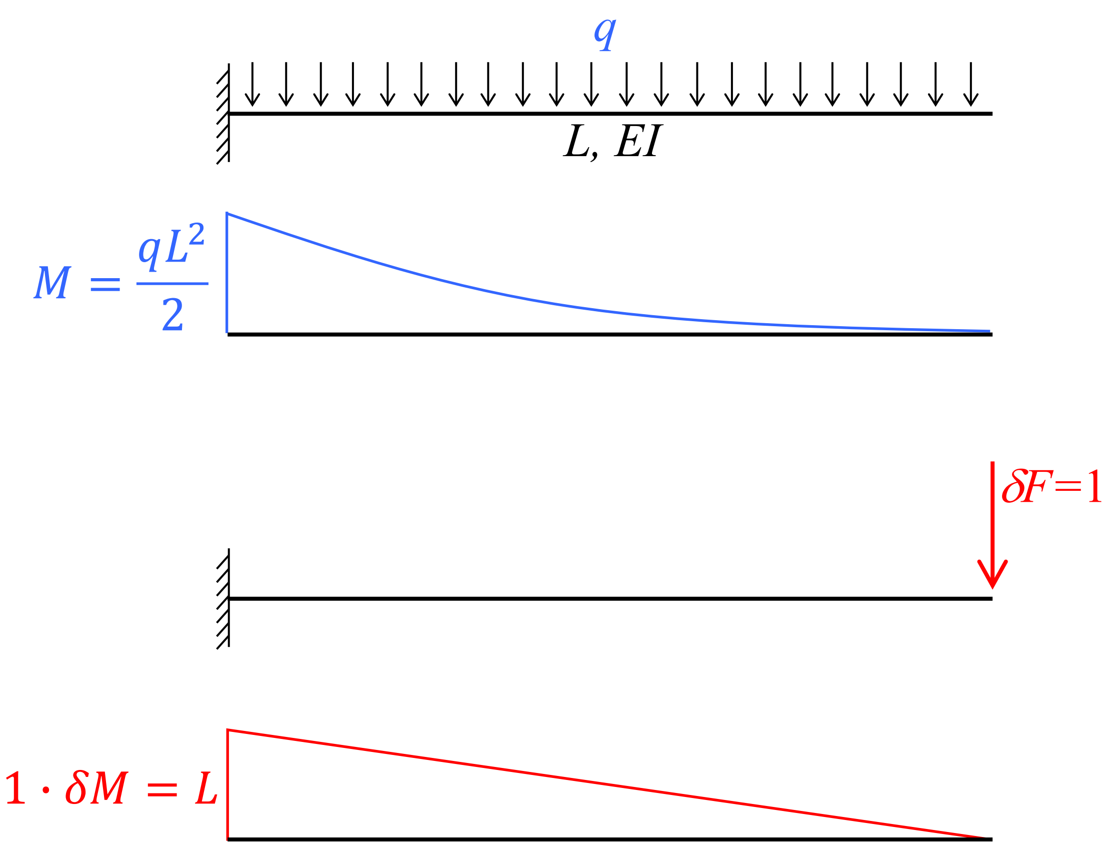


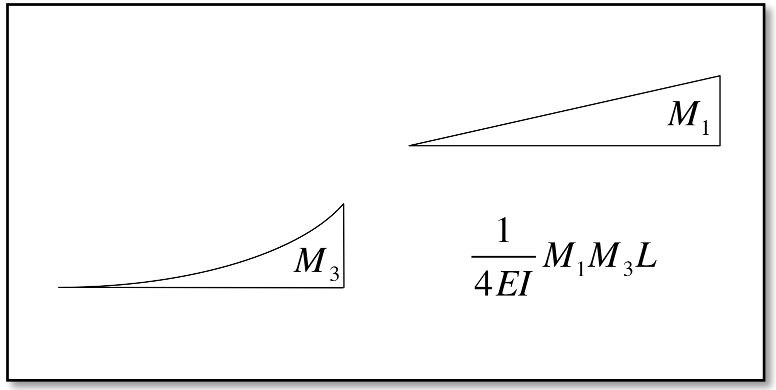
$$\color{red}{\delta F}\cdot\color{blue}{\Delta} = \int_0^L \frac{\color{red}{\delta M}\cdot\color{blue}{M}}{EI}\,\mathrm{d}x$$

Quick Integration

$\int_0^L \frac{\delta M \cdot M}{EI}\mathrm{d}x$				
	$\frac{1}{EI}M_1M_3L$	$\frac{1}{2EI}M_1M_3L$	$\frac{1}{2EI}(M_1+M_2)M_3L$	$\frac{1}{2EI}M_1M_3L$
	$\frac{1}{2EI}M_1M_3L$	$\frac{1}{3EI}M_1M_3L$	$\frac{1}{6EI}(M_1+2M_2)M_3L$	$\frac{1}{4EI}M_1M_3L$
	$\frac{1}{2EI}M_1M_3L$	$\frac{1}{6EI}M_1M_3L$	$\frac{1}{6EI}(2M_1+M_2)M_3L$	$\frac{1}{4EI}M_1M_3L$
	$\frac{1}{2EI}M_1(M_3+M_4)L$	$\frac{1}{6EI}M_1(M_3+2M_4)L$	$\frac{1}{6EI}M_1(2M_3+M_4)L$ $+\frac{1}{6EI}M_2(M_3+2M_4)L$	$\frac{1}{4EI}(M_1M_3+M_1M_4)L$
	$\frac{1}{2EI}M_1M_3L$	$\frac{1}{4EI}M_1M_3L$	$\frac{1}{4EI}(M_1M_3+M_2M_3)L$	$\frac{1}{3EI}M_1M_3L$
	$\frac{2}{3EI}M_1M_3L$	$\frac{1}{3EI}M_1M_3L$	$\frac{1}{3EI}(M_1+M_2)M_3L$	$\frac{5}{12EI}M_1M_3L$
	$\frac{1}{3EI}M_1M_3L$	$\frac{1}{4EI}M_1M_3L$	$\frac{1}{12EI}(M_1+3M_2)M_3L$	$\frac{7}{48EI}M_1M_3L$
	$\frac{2}{3EI}M_1M_3L$	$\frac{5}{12EI}M_1M_3L$	$\frac{1}{12EI}(3M_1+5M_2)M_3L$	

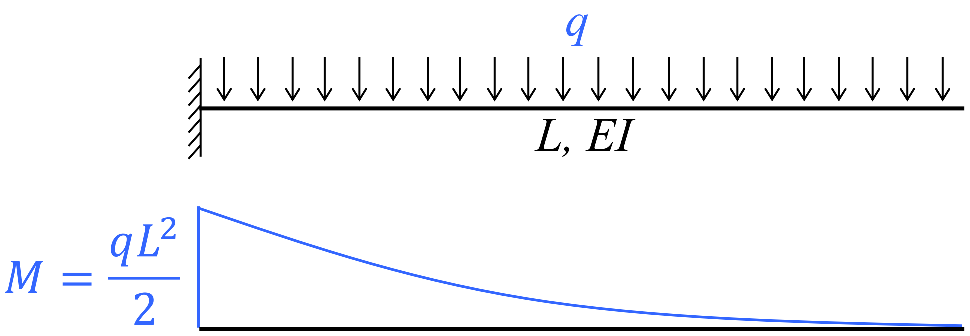
Example

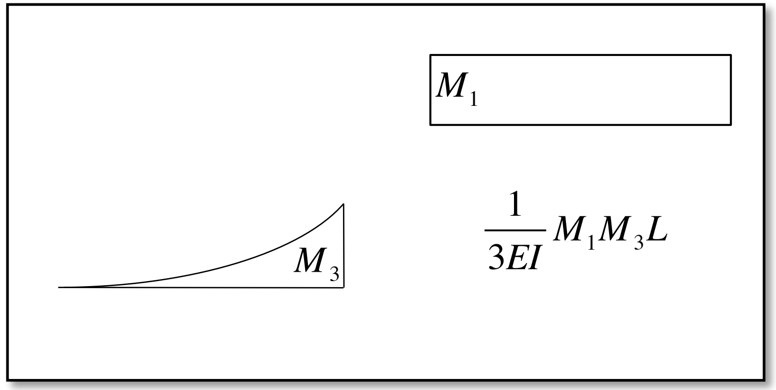


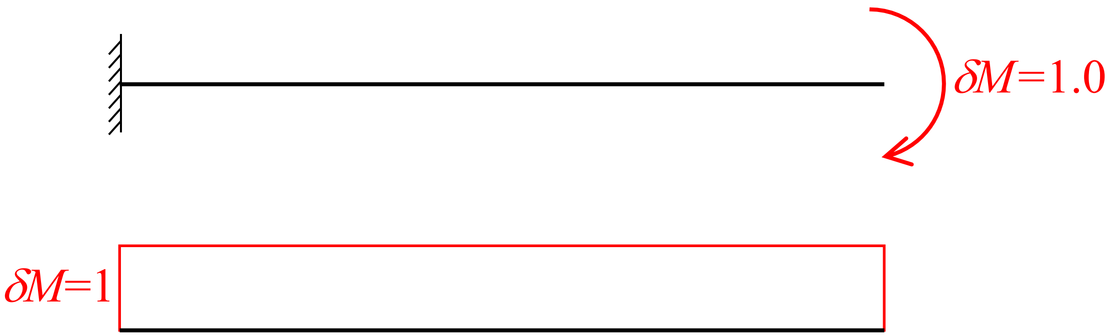


$$1 \cdot \Delta = \int_0^L \delta M \cdot \frac{M}{EI} dx = \frac{1}{4EI} \cdot \frac{qL^2}{2} \cdot L \cdot L = \frac{qL^4}{8EI}$$

Example

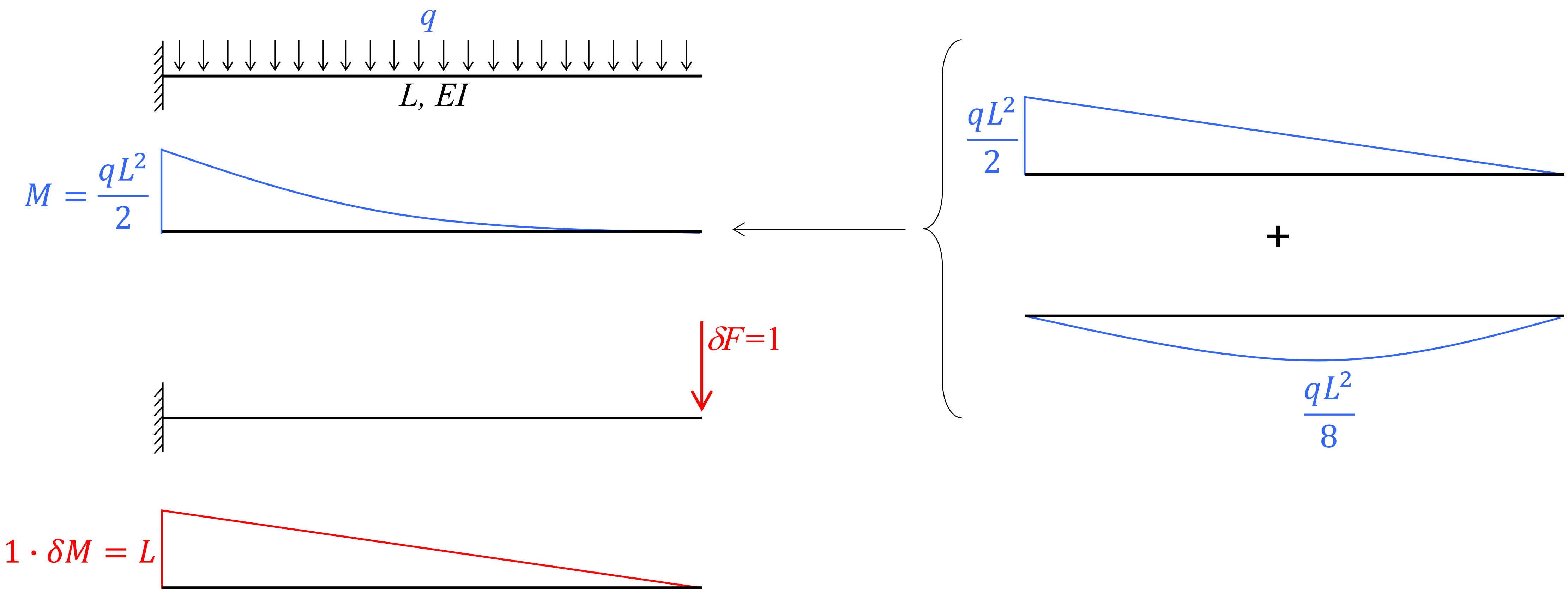




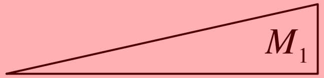
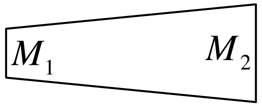
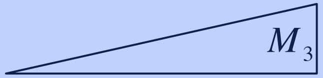
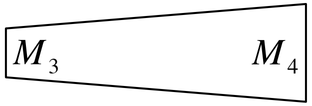
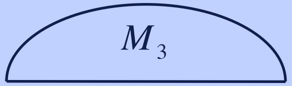
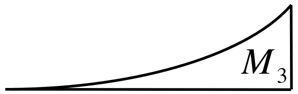
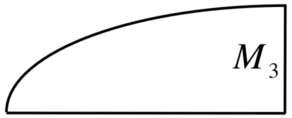


$$1 \cdot \theta = \int_0^L \delta M \cdot \frac{M}{EI} dx = \frac{1}{3EI} \cdot \frac{qL^2}{2} \cdot 1 \cdot L = \frac{qL^3}{6EI}$$

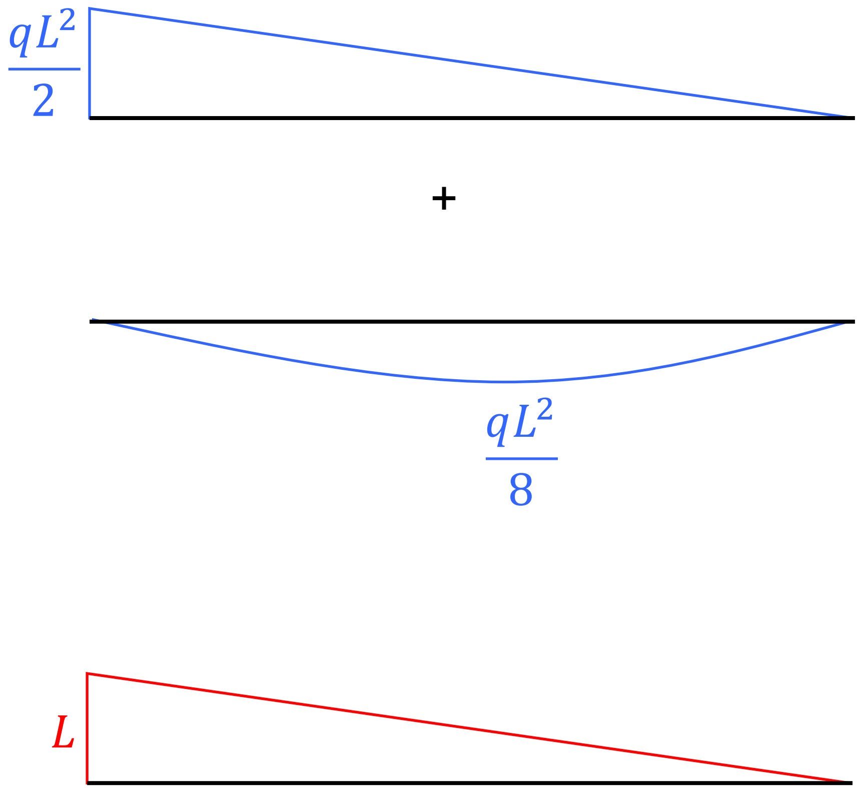
Basic Shapes



Quick Integration

$\int_0^L \frac{\delta M \cdot M}{EI}\mathrm{d}x$				
	$\frac{1}{EI}M_1M_3L$	$\frac{1}{2EI}M_1M_3L$	$\frac{1}{2EI}(M_1+M_2)M_3L$	$\frac{1}{2EI}M_1M_3L$
	$\frac{1}{2EI}M_1M_3L$	$\frac{1}{3EI}M_1M_3L$	$\frac{1}{6EI}(M_1+2M_2)M_3L$	$\frac{1}{4EI}M_1M_3L$
	$\frac{1}{2EI}M_1M_3L$	$\frac{1}{6EI}M_1M_3L$	$\frac{1}{6EI}(2M_1+M_2)M_3L$	$\frac{1}{4EI}M_1M_3L$
	$\frac{1}{2EI}M_1(M_3+M_4)L$	$\frac{1}{6EI}M_1(M_3+2M_4)L$	$\frac{1}{6EI}M_1(2M_3+M_4)L + \frac{1}{6EI}M_2(M_3+2M_4)L$	$\frac{1}{4EI}(M_1M_3+M_1M_4)L$
	$\frac{1}{2EI}M_1M_3L$	$\frac{1}{4EI}M_1M_3L$	$\frac{1}{4EI}(M_1M_3+M_2M_3)L$	$\frac{1}{3EI}M_1M_3L$
	$\frac{2}{3EI}M_1M_3L$	$\frac{1}{3EI}M_1M_3L$	$\frac{1}{3EI}(M_1+M_2)M_3L$	$\frac{5}{12EI}M_1M_3L$
	$\frac{1}{3EI}M_1M_3L$	$\frac{1}{4EI}M_1M_3L$	$\frac{1}{12EI}(M_1+3M_2)M_3L$	$\frac{7}{48EI}M_1M_3L$
	$\frac{2}{3EI}M_1M_3L$	$\frac{5}{12EI}M_1M_3L$	$\frac{1}{12EI}(3M_1+5M_2)M_3L$	

Same Result

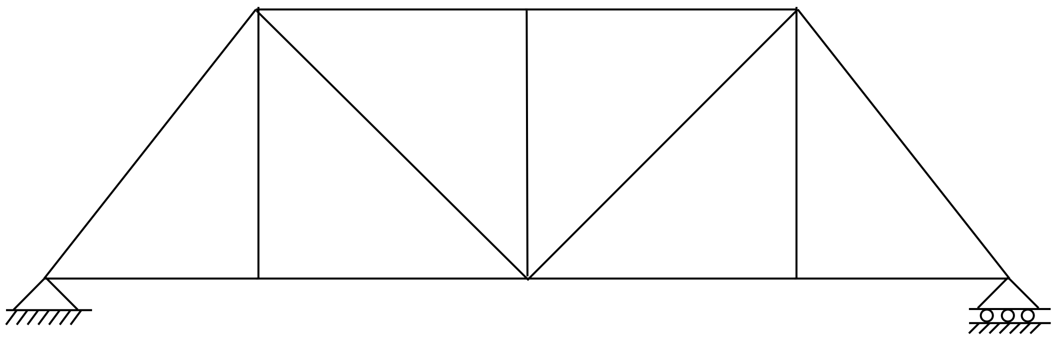


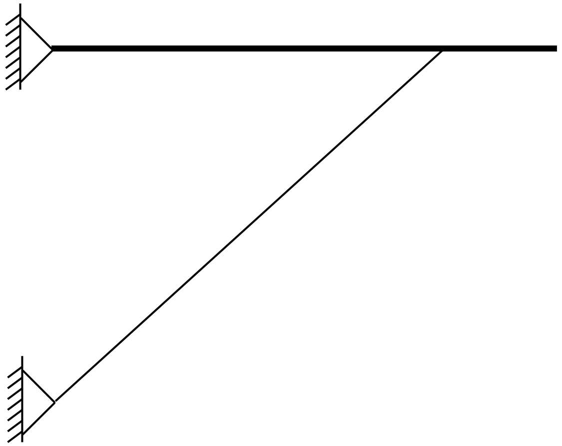
$$1 \cdot \Delta = \frac{1}{3EI} \cdot \frac{qL^2}{2} \cdot L \cdot L - \frac{1}{3EI} \cdot \frac{qL^2}{8} \cdot L \cdot L = \frac{qL^4}{8EI}$$

Axial & Shear Deformations?

Axial Flexural Shear

$$\delta F \cdot \Delta = \sum_{\substack{\text{Sum over}\\ \text{all members}}} \left(\frac{\delta N \cdot N \cdot L}{EA} + \int_0^L \frac{\delta M \cdot M}{EI}\,\mathrm{d}x + \int_0^L \frac{\delta V \cdot V}{G \cdot A_v}\,\mathrm{d}x \right)$$





Settlements, Temperature Change

$$\begin{pmatrix} \delta F \cdot \Delta \\ +\delta F_{S1} \cdot \Delta_{S1} \\ +\delta F_{S2} \cdot \Delta_{S2} \\ +\cdots \end{pmatrix} = \sum_{\substack{\text{Sum over} \\ \text{all members}}} \begin{pmatrix} \delta N \cdot \left(\dfrac{N \cdot L}{EA} + \alpha \cdot \Delta T \cdot L + \Delta L_{\text{fab. error}} \right) \\ + \displaystyle\int_0^L \delta M \cdot \left(\dfrac{M}{EI} \pm \alpha \cdot \dfrac{\left| \Delta T_{top} - \Delta T_{bottom} \right|}{h} \right) dx \\ + \displaystyle\int_0^L \dfrac{\delta V \cdot V}{G \cdot A_v} \, \mathrm{d}x \end{pmatrix}$$

Signs

Negative when virtual force is tension with real shortening, and when virtual force is compression with real elongation

Negative when settlement is in the opposite direction of the virtual support reaction force

$$\begin{pmatrix} \delta F \cdot \Delta \\ +\delta F_{S1} \cdot \Delta_{S1} \\ +\delta F_{S2} \cdot \Delta_{S2} \\ +\cdots \end{pmatrix} = \sum_{\substack{\text{Sum over} \\ \text{all members}}} \begin{pmatrix} \delta N \cdot \left(\dfrac{N \cdot L}{EA} + \alpha \cdot \Delta T \cdot L + \Delta L_{\text{fab. error}} \right) \\ + \displaystyle\int_0^L \delta M \cdot \left(\dfrac{M}{EI} \pm \alpha \cdot \dfrac{\left|\Delta T_{top} - \Delta T_{bottom}\right|}{h} \right) dx \\ + \displaystyle\int_0^L \dfrac{\delta V \cdot V}{G \cdot A_v} \, \mathrm{d}x \end{pmatrix}$$

Negative when tension on different sides

Negative when shear forces have opposite sign

More lectures:

Terje's Toobox:

terje.civil.ubc.ca