A short course on

The Finite Element Method

This video:

The Finite Element Method for Truss and Beam Elements

Terje's Toolbox is freely available at terje.civil.ubc.ca
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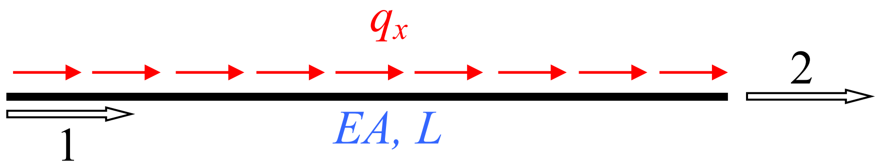
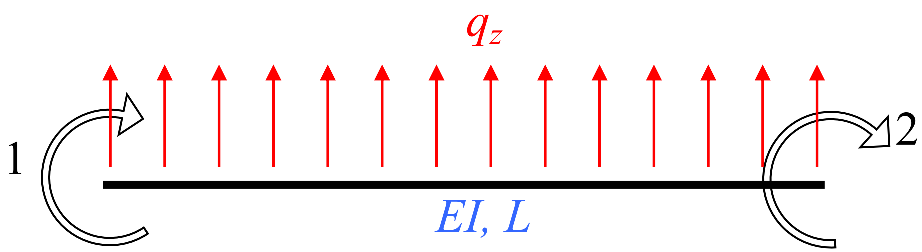
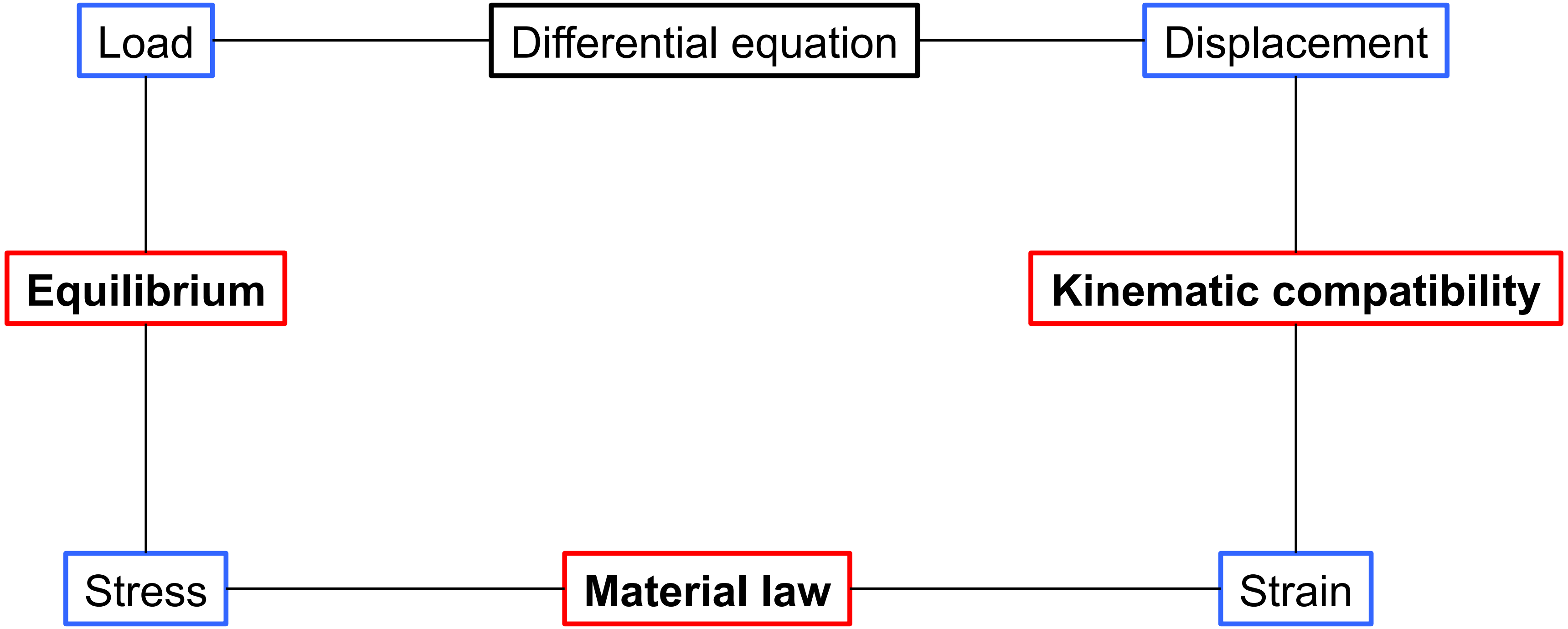
Elements

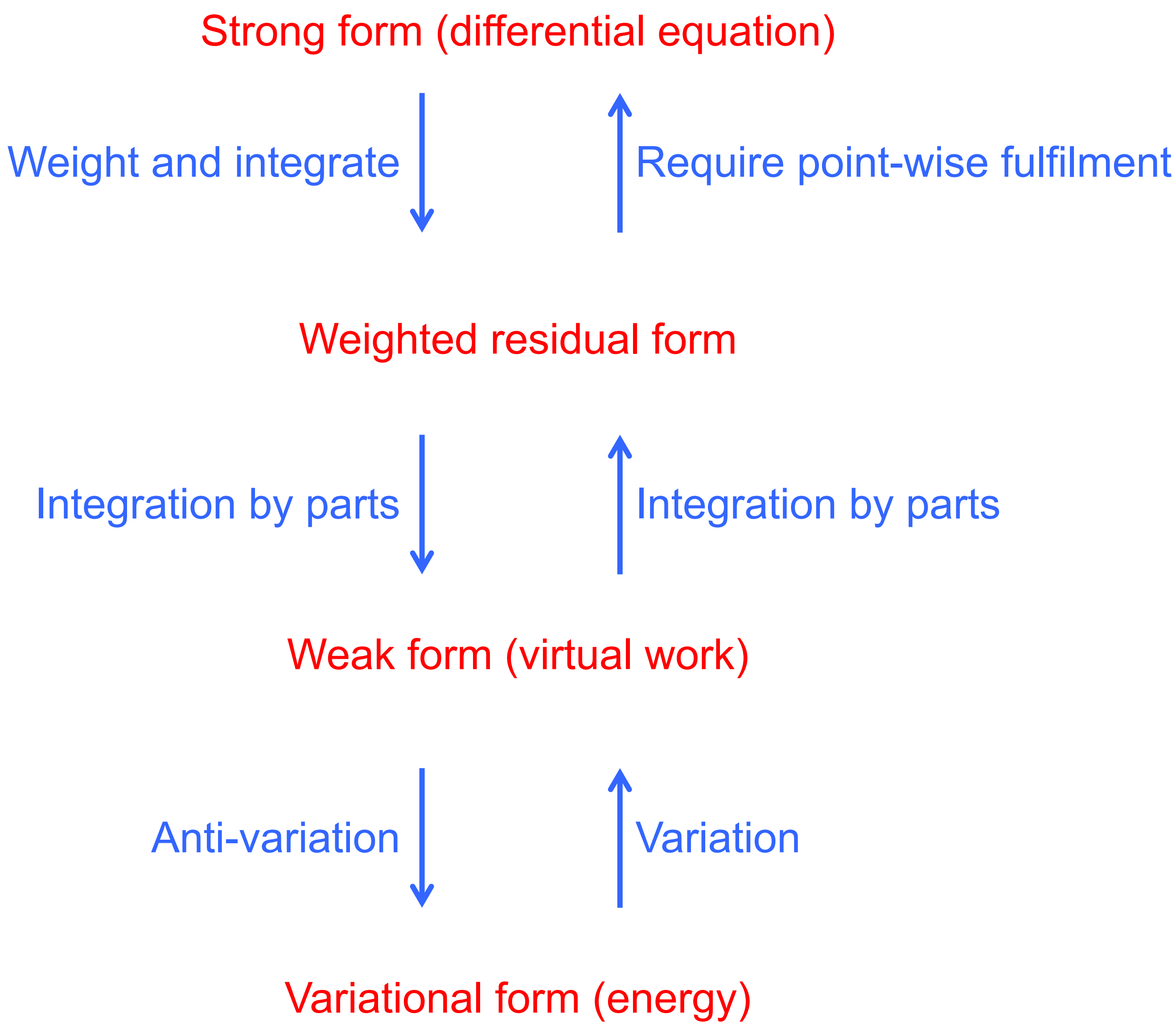




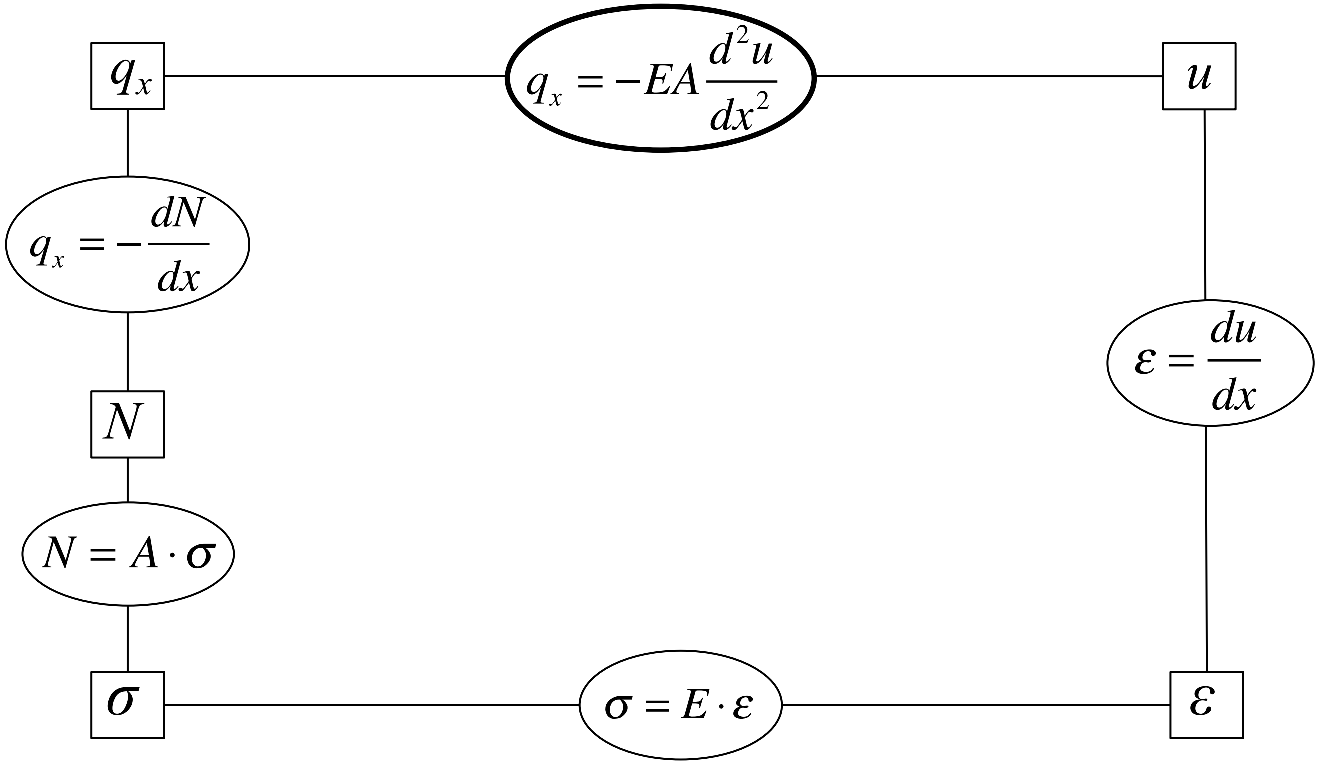
Boundary Value Problems



Forms of the BVP



Truss Member



Forms of the Truss BVP

$$\frac{du}{dx} \equiv u'$$

$$\frac{d^2u}{dx^2} \equiv u''$$

Strong form (differential equation):

$$EA \cdot u'' + q_x = 0$$

Weight and integrate ↓ ↑ Require point-wise fulfilment

Weighted residual form:

$$\int_0^L (EA \cdot u'' + q_x) \cdot \delta u \, dx = 0$$

Integration by parts ↓ ↑ Integration by parts

Weak form (virtual work):

$$\int_0^L (-EA \cdot u' \cdot \delta u' + q_x \cdot \delta u) \, dx = 0$$

Anti-variation ↓ ↑ Variation

Variational form (energy):

$$\int_0^L \left(-\frac{1}{2} \cdot EA \cdot (u')^2 + q_x \cdot u\right) dx = 0$$

Weak Form = Virtual Work

Virtual work: $\delta W_{\text{int}} = \delta W_{\text{ext}}$

Principle of virtual displacements: $\int_V \sigma \cdot \delta\varepsilon \, dV = \int_0^L q_x \cdot \delta u \, dx$

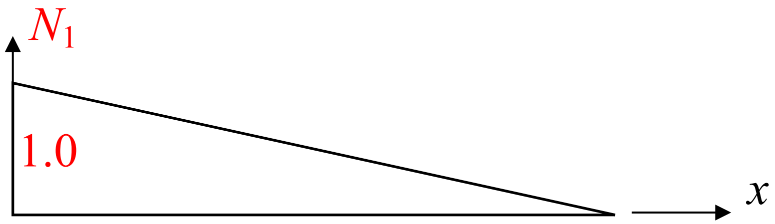
$\sigma = E \cdot \varepsilon$ Substitute material law: $\int_V E \cdot \varepsilon \cdot \delta\varepsilon \, dV = \int_0^L q_x \cdot \delta u \, dx$

$\varepsilon = \frac{du}{dx}$ Substitute kinematic compatibility: $\int_V E \cdot u' \cdot \delta u' \, dV = \int_0^L EA \cdot u' \cdot \delta u' \, dx = \int_0^L q_x \cdot \delta u \, dx$

Weak form from previous slide: $\int_0^L (-EA \cdot u' \cdot \delta u' + q_x \cdot \delta u) \, dx = 0$

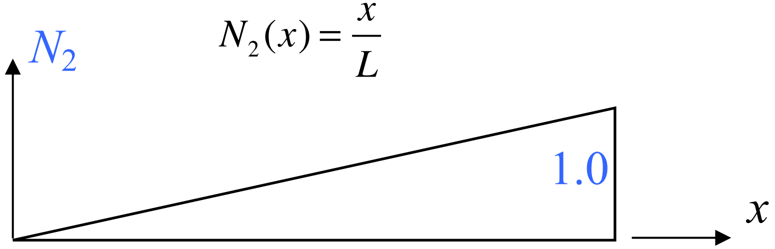
Discretization

$$\int_0^L \left(-EA \cdot u' \cdot \delta u' + q_x \cdot \delta u\right) dx = 0$$



$$N_1(x) = 1 - \frac{x}{L}$$

$$N_2(x) = \frac{x}{L}$$



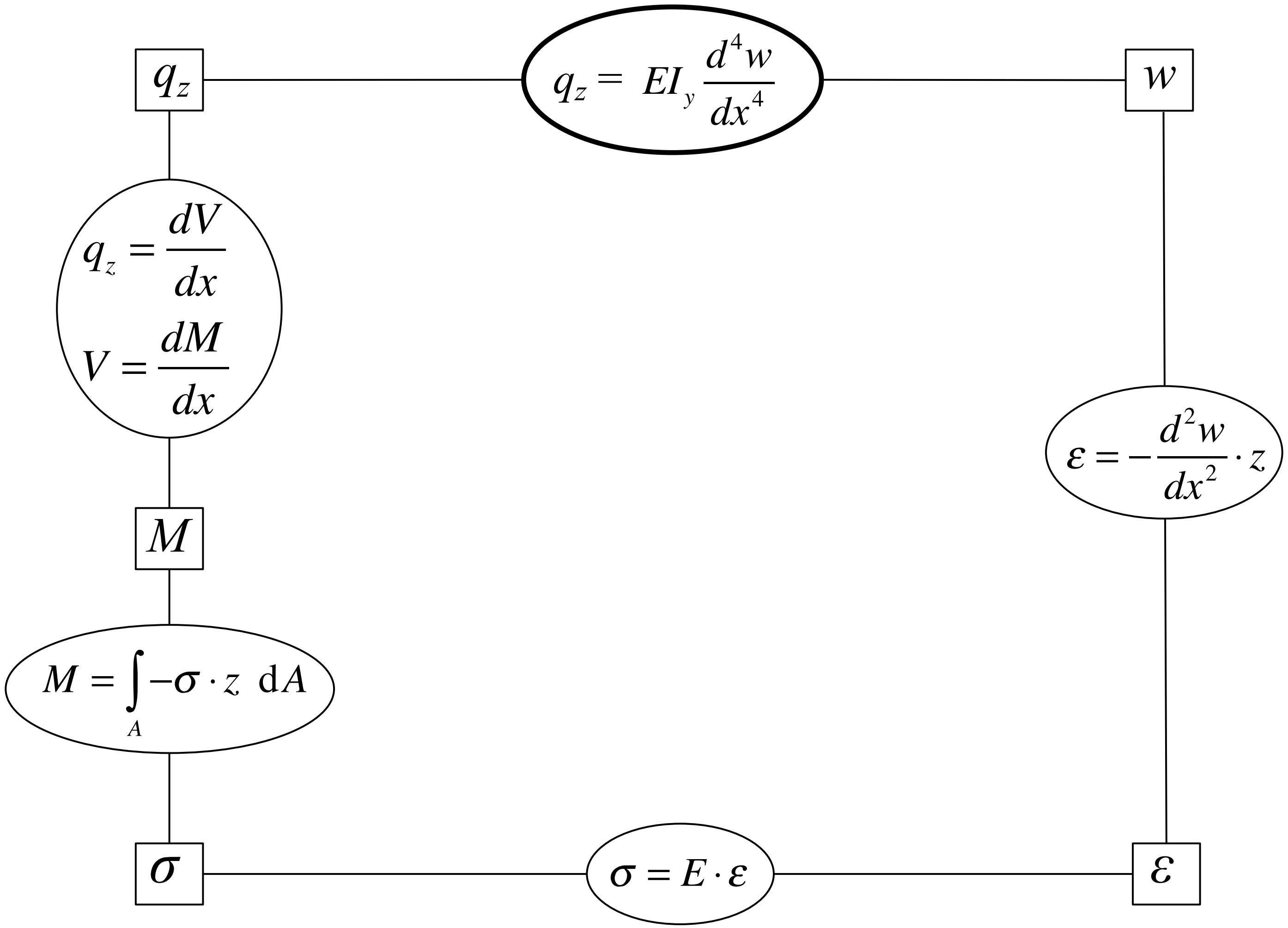
$$u(x) = \mathbf{Nu} = \begin{bmatrix} N_1(x) & N_2(x) \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$-\int_0^L EA\left(\mathbf{N}'\mathbf{u}\right)\left(\mathbf{N}'\delta\mathbf{u}\right)dx + \int_0^L q_x\left(\mathbf{N}\delta\mathbf{u}\right)dx = 0$$

$$\delta\mathbf{u}^T\left(-\left(\int_0^L EA \cdot \mathbf{N}'^T\,\mathbf{N}'\,dx\right)\mathbf{u} + \int_0^L q_x \mathbf{N}^T\,dx\right) = 0$$

$$\underbrace{\left(\int_0^L EA \cdot \mathbf{N}'^T\,\mathbf{N}'\,dx\right)}_{\mathbf{K}}\mathbf{u} = \underbrace{\int_0^L q_x \mathbf{N}^T\,dx}_{\mathbf{F}} \quad \Rightarrow \quad \begin{bmatrix} \frac{EA}{L} & -\frac{EA}{L} \\ -\frac{EA}{L} & \frac{EA}{L} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} \frac{q_x L}{2} \\ \frac{q_x L}{2} \end{Bmatrix}$$

Beam Bending



Forms of the BVP

$$\frac{d^2w}{dx^2} \equiv w''$$

$$\frac{d^4w}{dx^4} \equiv w''''$$

Strong form (differential equation): $EI \cdot w'''' - q_z = 0$

↓ Weight and integrate ↑ Require point-wise fulfilment

Weighted residual form: $\int_0^L (EI \cdot w'''' - q_z) \cdot \delta w \, dx = 0$

↓ Integration by parts ↑ Integration by parts

Weak form (virtual work): $\int_0^L (EI \cdot w'' \cdot \delta w'' - q_z \cdot \delta w) \, dx = 0$

↓ Anti-variation ↑ Variation

Variational form (energy): $\int_0^L \left(\frac{1}{2} \cdot EI \cdot (w'')^2 - q_z \cdot w\right) dx = 0$

Weak Form = Virtual Work

Virtual work: $\delta W_{\text{int}} = \delta W_{\text{ext}}$

Principle of virtual displacements: $$\int_V \sigma \cdot \delta\varepsilon \, dx = \int_0^L q_z \cdot \delta w \, dx$$

$\sigma = E \cdot \varepsilon$

Substitute material law: $$\int_V E \cdot \varepsilon \cdot \delta\varepsilon \, dx = \int_0^L q_z \cdot \delta w \, dx$$

$\varepsilon = -\frac{d^2 w}{dx^2} \cdot z$

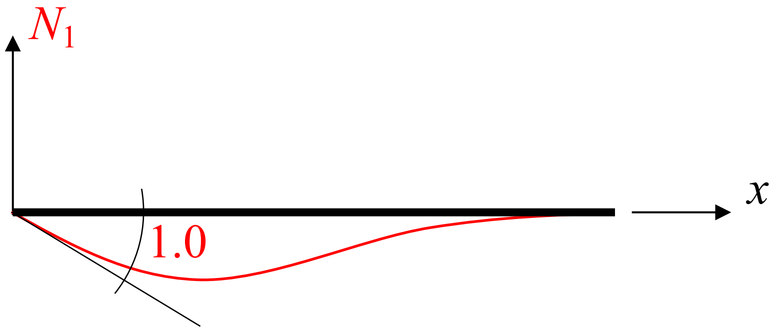
Substitute kinematic compatibility: $$\int_V E \cdot w'' \cdot \delta w'' \cdot z^2 \, dx = \int_0^L EI \cdot w'' \cdot \delta w'' \, dx = \int_0^L q_z \cdot \delta w \, dx$$

Weak form from previous slide: $$\int_0^L (EI \cdot w'' \cdot \delta w'' - q_z \cdot \delta w) \, dx = 0$$

Discretization

$$\int_0^L (EI \cdot w'' \cdot \delta w'' - q_z \cdot \delta w)\, dx = 0$$

$$w(x) = \mathbf{N}\mathbf{u} = \begin{bmatrix} N_1(x) & N_2(x) \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

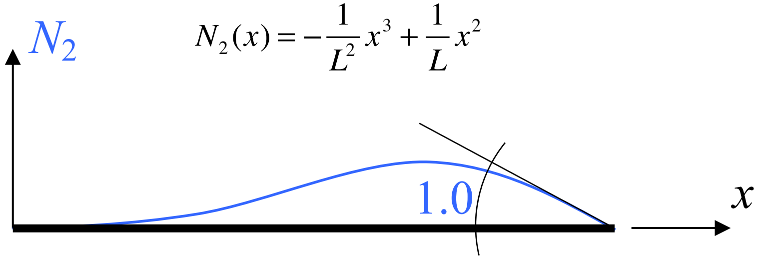


$$\int_0^L EI \cdot \left(\mathbf{N}''\mathbf{u}\right) \cdot \left(\mathbf{N}''\delta\mathbf{u}\right)\ \mathrm{d}x - \int_0^L q_z\ \left(\mathbf{N}\delta\mathbf{u}\right)\ \mathrm{d}x = 0$$

$$N_1(x) = -\frac{1}{L^2}x^3 + \frac{2}{L}x^2 - x$$

$$N_2(x) = -\frac{1}{L^2}x^3 + \frac{1}{L}x^2$$

$$\delta\mathbf{u}^T\left(\left[\int_0^L EI \cdot \mathbf{N}''^T\mathbf{N}''\ \mathrm{d}x\right]\mathbf{u} - \int_0^L q_z\ \mathbf{N}^T\ \mathrm{d}x\right) = 0$$



$$\underbrace{\left[\int_0^L EI \cdot \mathbf{N}''^T\,\mathbf{N}''\,dx\right]}_{\text{Stiffness matrix, }\mathbf{K}}\mathbf{u} = \underbrace{\int_0^L q_z \mathbf{N}\,dx}_{\text{Load vector, }\mathbf{F}} \quad \Rightarrow \quad \begin{bmatrix} \frac{4EI}{L} & \frac{2EI}{L} \\ \frac{2EI}{L} & \frac{4EI}{L} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} -\frac{q_z L^2}{12} \\ \frac{q_z L^2}{12} \end{Bmatrix}$$

More lectures:

Terje’s Toobox:

terje.civil.ubc.ca