A short course on

Cross-section Analysis

This video:

Stress Functions for the Calculation of Cross-section Constant and Stress in Saint Venant Torsion

Terje's Toolbox is freely available at terje.civil.ubc.ca
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Scope

y_o, z_o = centroid coordinates

y_{sc}, z_{sc} = shear centre coordinates

A = cross-section area

I_y , I_z = moments of inertia

I_{yz} = product of inertia

θ = orientation of principal axes

J = Saint Venant torsion constant

Ω = omega diagram

C_w = warping torsion constant

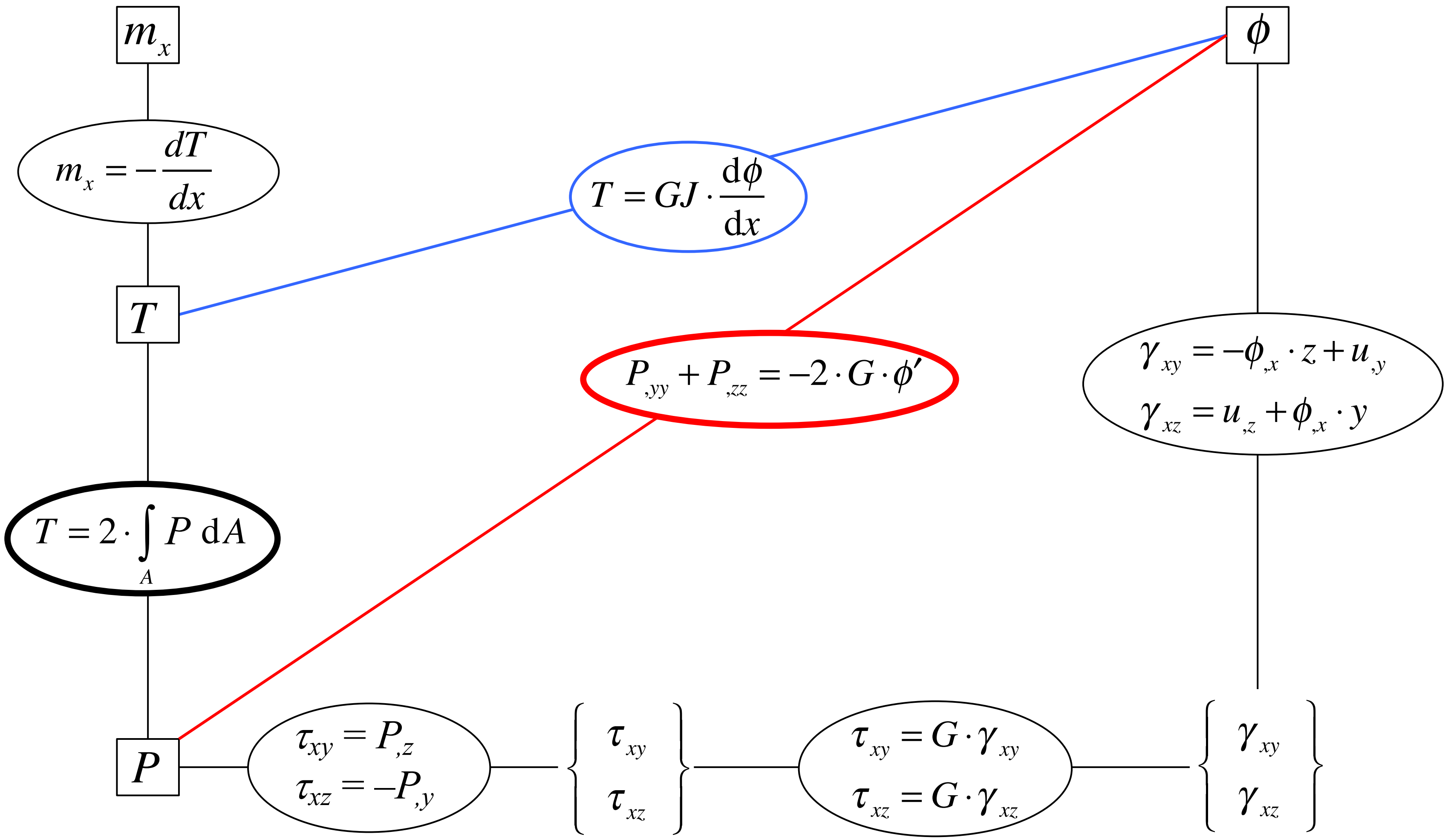
A_{vy}, A_{vz} = shear area

σ = axial stress

τ = shear stress

q_s = shear flow

Summary


m_x
$m_x = -\frac{dT}{dx}$
T
$T = GJ \cdot \frac{\mathrm{d}\phi}{\mathrm{d}x}$
ϕ
$P_{,yy} + P_{,zz} = -2 \cdot G \cdot \phi'$
$\gamma_{xy} = -\phi_{,x} \cdot z + u_{,y}$
$\gamma_{xz} = u_{,z} + \phi_{,x} \cdot y$
$T = 2 \cdot \int_A P \, \mathrm{d}A$
P
$\tau_{xy} = P_{,z}$
$\tau_{xz} = -P_{,y}$
$\left\{ \begin{matrix} \tau_{xy} \\ \tau_{xz} \end{matrix} \right\}$
$\tau_{xy} = G \cdot \gamma_{xy}$
$\tau_{xz} = G \cdot \gamma_{xz}$
$\left\{ \begin{matrix} \gamma_{xy} \\ \gamma_{xz} \end{matrix} \right\}$

Cross-section constant, J

$$T = GJ \cdot \frac{\mathrm{d}\phi}{\mathrm{d}x}$$

$$T = 2 \cdot \int_A P \,\mathrm{d}A$$

$$P_{,yy} + P_{,zz} = -2 \cdot G \cdot \phi'$$

$$\underbrace{\left(2 \cdot \int_A P \cdot \mathrm{d}A\right)}_{T} = GJ \cdot \underbrace{\left(-\frac{P_{,yy} + P_{,zz}}{2 \cdot G}\right)}_{\phi_{,x}}$$

$$J = -\frac{4 \cdot V}{\nabla^2 P}$$

Alternative J

$$\int_{A_\Gamma} P_{,yy} + P_{,zz}\, dA = -2 \cdot G \cdot \phi_{,x} \int_{A_\Gamma} dA$$

$$\int_{A_\Gamma} P_{,yy} + P_{,zz}\, dA = \oint_\Gamma \left(P_{,y} dz - P_{,z} dy \right)$$

$$\oint_\Gamma \left(P_{,y} dz - P_{,z} dy \right) = \oint_\Gamma \left(\tau_{xz} dz + \tau_{xy} dy \right) = \oint_\Gamma \left(\tau_{xz} \frac{dz}{ds} + \tau_{xy} \frac{dy}{ds} \right) ds = \oint_\Gamma \tau_{xs}\, ds$$

$$T = 2 \cdot \int_A P \,\mathrm{d}A$$

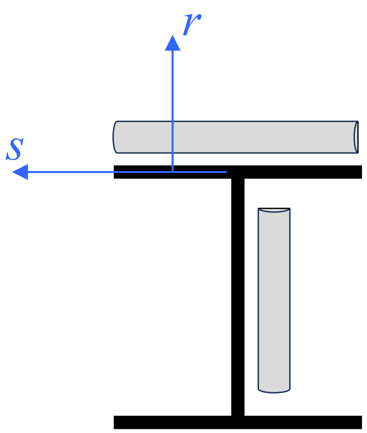
$$T = GJ \cdot \frac{\mathrm{d}\phi}{\mathrm{d}x}$$

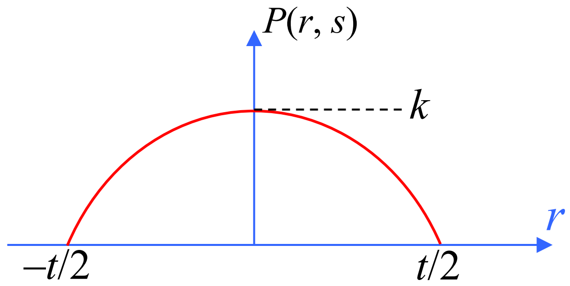
$$\oint_\Gamma \tau_{xs}\, ds = -2 \cdot G \cdot \phi' \cdot A_\Gamma$$

$$J = -\frac{4 \cdot V \cdot A_\Gamma}{\oint_\Gamma \tau_{xs}\, ds}$$

See Bredt's formula later…

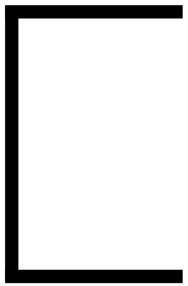
Thin-walled, Open





$$P(r,s) = k \cdot \left(1 - 4 \cdot \frac{r^2}{t^2}\right)$$

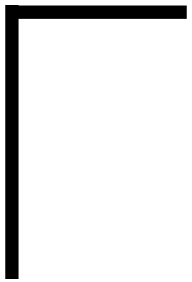
$$J = -\frac{4 \cdot V}{\nabla^2 P}$$



$$\left.\begin{aligned} V \equiv \int_A P \cdot \mathrm{d}A = \int_{-t/2}^{t/2} \int_{-b/2}^{b/2} k\left(1 - 4 \cdot \frac{r^2}{t^2}\right) ds\, dr = \frac{2}{3} \cdot k \cdot t \cdot b \\ \nabla^2 P \equiv P_{,ss} + P_{,rr} = -\frac{8k}{t^2} \end{aligned}\right\} \Rightarrow \quad J = \frac{1}{3} \cdot t^3 \cdot b$$

$$\tau_{xs} = P_{,r} = \frac{8k}{t^2} \cdot r$$

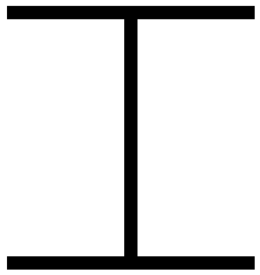
$$\tau_{xr} = -P_{,s} = 0$$

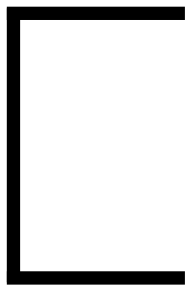


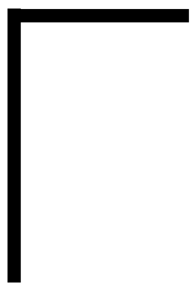
$$T = 2 \cdot \int_A P \cdot \mathrm{d}A = 2 \cdot V$$

$$\tau_{xs} = \left(\frac{8 \cdot r}{t_i^2}\right) \cdot k = \left(\frac{8 \cdot r}{t_i^2}\right) \cdot \left(\frac{3 \cdot T}{4 \cdot t \cdot b}\right)$$

Cross-sections with Several Parts







$$T = \sum T_i$$

$$T = \sum \left(GJ_i \cdot \frac{\mathrm{d}\phi}{\mathrm{d}x} \right) = \frac{\mathrm{d}\phi}{\mathrm{d}x} \cdot \sum (GJ_i) \equiv \frac{\mathrm{d}\phi}{\mathrm{d}x} \cdot GJ$$

$$T_i = GJ_i \cdot \frac{\mathrm{d}\phi_i}{\mathrm{d}x} = \frac{GJ_i}{\sum (GJ_i)} \cdot T$$

$$T_i = 2 \cdot \int_A P \cdot \mathrm{d}A = 2 \cdot V_i$$

$$V_i = \frac{2}{3} \cdot k_i \cdot t_i \cdot b_i$$

$$\tau_{xs} = \left(\frac{8 \cdot r}{t_i^2} \right) \cdot k = \left(\frac{8 \cdot r}{t_i^2} \right) \cdot \left(\frac{3 \cdot T_i}{4 \cdot t_i \cdot b_i} \right)$$

J_i

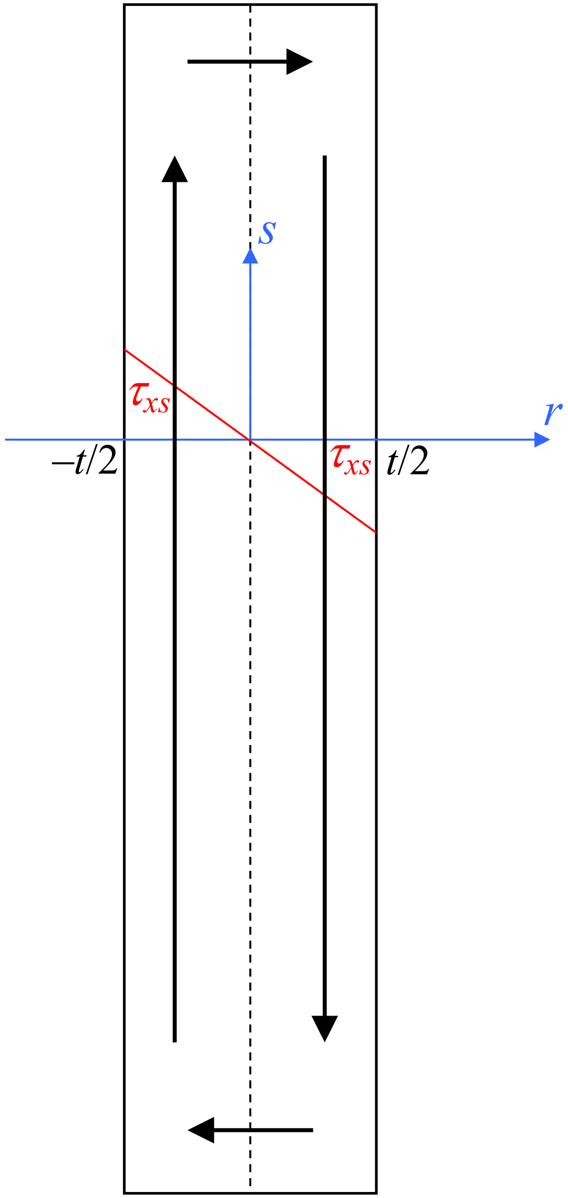
$$GJ = \Sigma GJ_i$$

$$T_i = \frac{GJ_i}{\sum (GJ_i)} \cdot T$$

$$V_i = T_i / 2 \rightarrow k_i$$

$$\tau_{xs} = \left(\frac{8 \cdot r}{t_i^2} \right) \cdot k_i$$

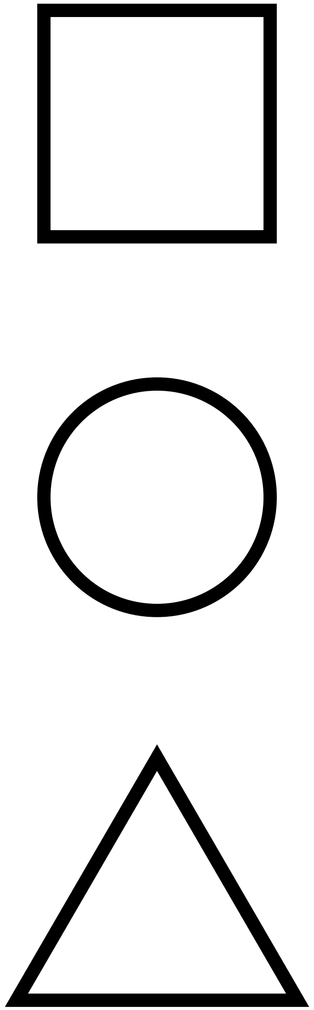
Checking the Torque

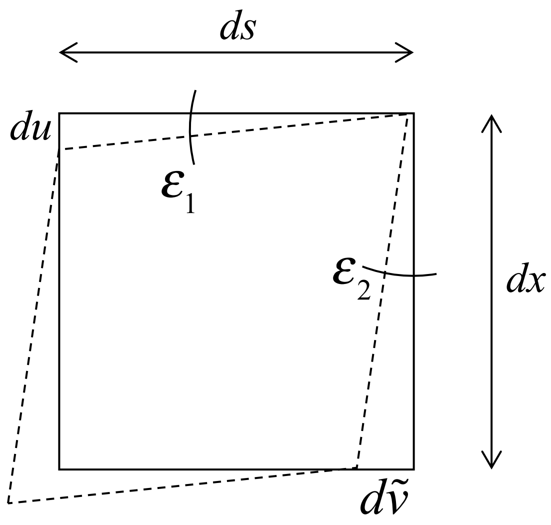


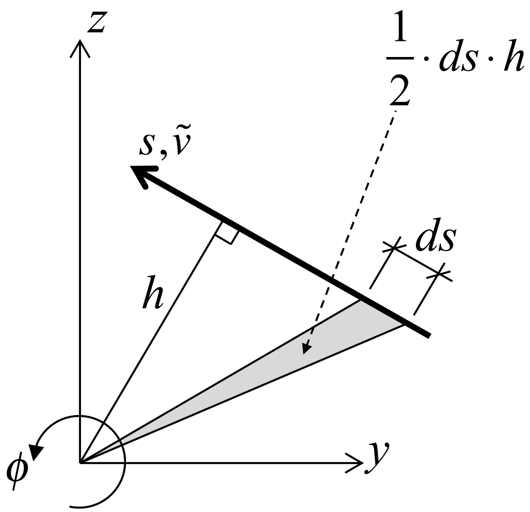
$$\tau_{xs} = \frac{8 \cdot \frac{t}{2}}{t^2} \cdot \frac{3 \cdot T}{4 \cdot t \cdot b} = \frac{3 \cdot T}{t^2 \cdot b}$$

$$T = \left(\frac{1}{2} \cdot \tau_{xs} \cdot \frac{t}{2}\right) \cdot b \cdot \frac{2}{3} \cdot t = \frac{T}{2}$$

Closed Cross-sections







$$\oint_{\text{Closed curve}} du = 0$$

$$\gamma_{xs} = \varepsilon_{xs,1} + \varepsilon_{xs,2} = \frac{du}{ds} + \frac{d\tilde{v}}{dx}$$

$$du = \left(\frac{\tau_{xs}}{G} - \frac{d\tilde{v}}{dx}\right)\cdot ds$$

$$\oint_{\text{Closed curve}} \left(\frac{\tau_{xs}}{G} - \frac{d\tilde{v}}{dx}\right) ds = 0$$

$$\oint_{\text{Closed curve}} \left(\frac{\tau_{xs}}{G} - \frac{d\phi}{dx}\cdot h\right) ds = \oint_{\text{Closed curve}} \frac{\tau_{xs}}{G} ds - \frac{d\phi}{dx}\cdot \oint_{\text{Closed curve}} h\, ds = 0$$

$$\oint_{\text{Closed curve}} h \cdot ds \equiv 2 \cdot A$$

$$\oint_{\text{Closed curve}} \frac{\tau_{xs}}{G} ds - \frac{d\phi}{dx}\cdot 2 \cdot A = 0$$

$$\oint_{\text{Closed curve}} \tau_{xs}\, ds = \frac{T}{J}\cdot 2 \cdot A$$

Thin-walled, One Cell

$$P(s,r) = K \cdot \left(\frac{1}{2} + \frac{r}{t}\right) + M \cdot \left(1 - \left(\frac{2r}{t}\right)^2\right)$$

$$\tau_{xs} = \frac{\partial P}{\partial r} = \frac{K}{t} - \frac{8Mr}{t^2}$$

$$K \cdot \oint_{\Gamma_m} \frac{1}{t} \mathrm{d}s = \frac{T}{J} \cdot 2 \cdot A_m$$

$$K \cdot \oint_{\Gamma_i} \frac{1}{t} \mathrm{d}s - 4M \cdot \oint_{\Gamma_i} \frac{1}{t} \mathrm{d}s = \frac{T}{J} \cdot 2 \cdot A_i$$

$$M = \frac{K}{4} \cdot \left(1 - \frac{A_i}{A_m} \cdot \frac{\oint_{\Gamma_m} \frac{1}{t} \mathrm{d}s}{\oint_{\Gamma_i} \frac{1}{t} \mathrm{d}s}\right)$$

$$P(s,r) = K \cdot \left(\frac{1}{2} + \frac{r}{t}\right)$$

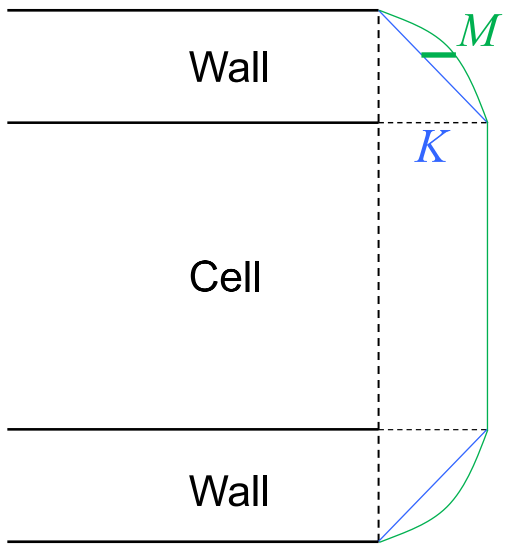
$$\tau_{xs} = P_{,r} = \frac{K}{t}$$

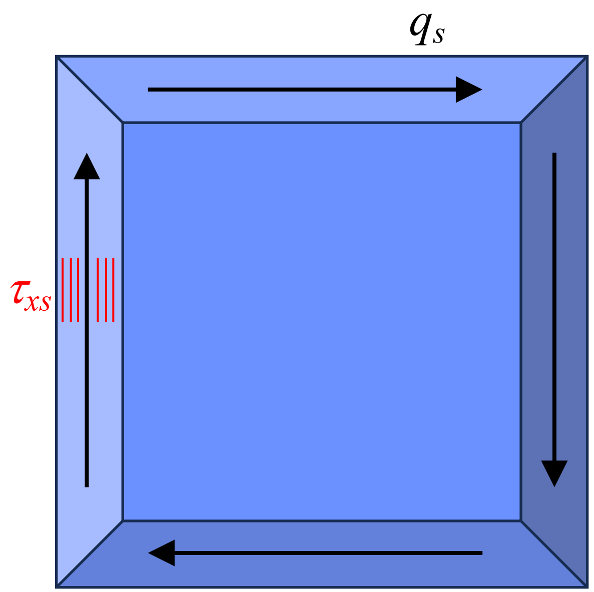
$$q_s = \tau_{xs} \cdot t = K$$

$$T = 2 \cdot \int_A P \cdot \mathrm{d}A = 2 \cdot K \cdot A_m \quad \Rightarrow \quad K = \frac{T}{2 \cdot A_m}$$

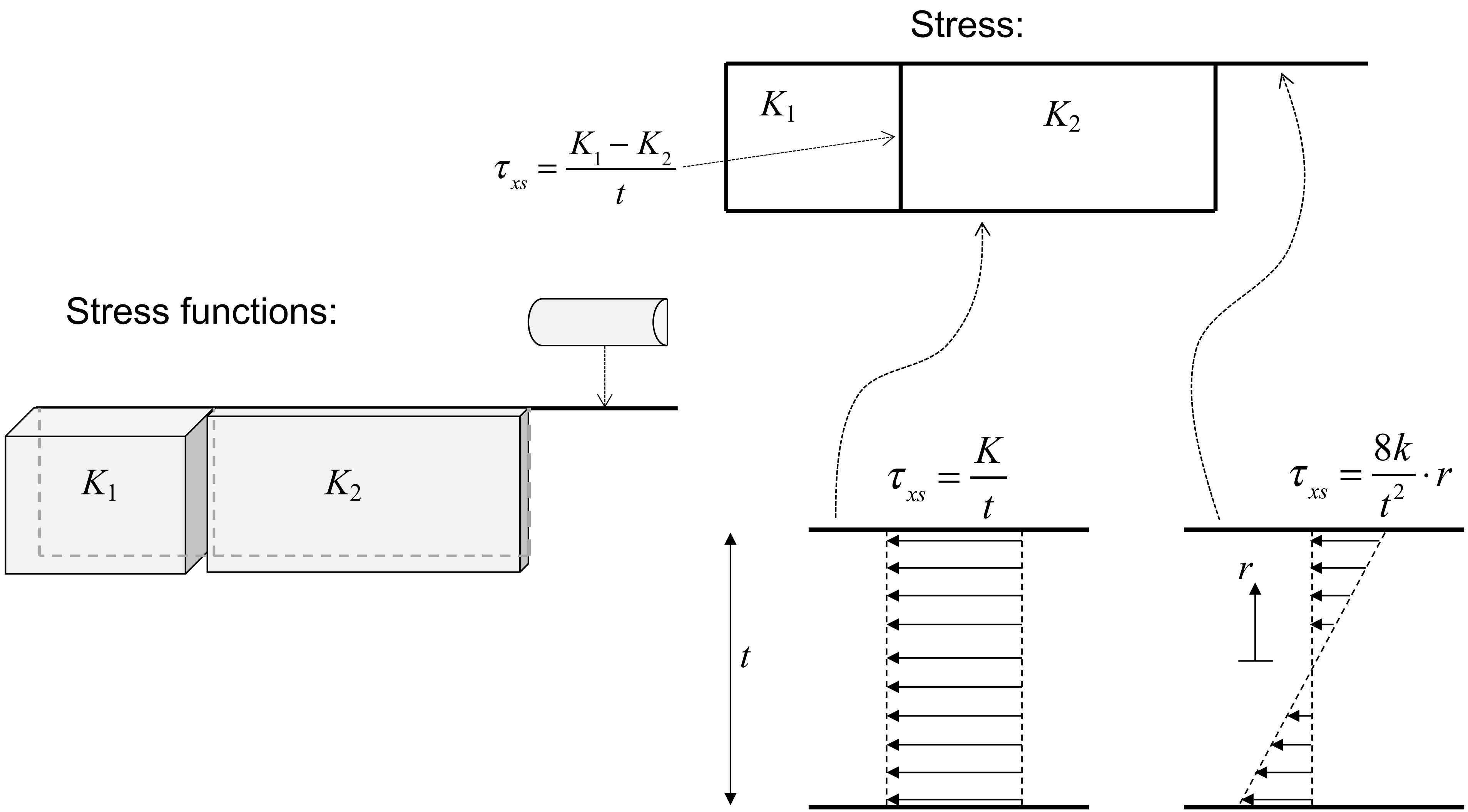
Bredt:

$$J = \frac{4 \cdot A_m^2}{\oint_{\Gamma_m} \frac{1}{t} \mathrm{d}s}$$





Visualization



More lectures:

Terje’s Toobox:

terje.civil.ubc.ca