A short course on

Probabilities and Random Variables

This video:

Individual Random Variables

Terje's Toolbox is freely available at terje.civil.ubc.ca
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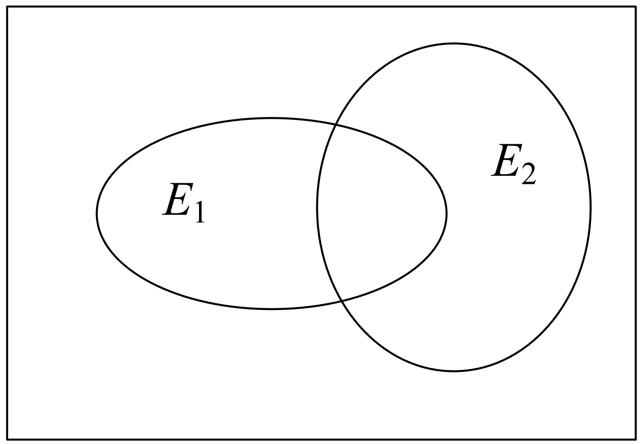
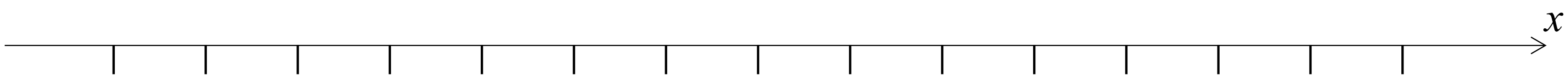
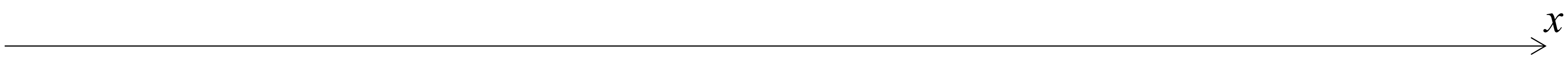
Outcome Space



Discrete:

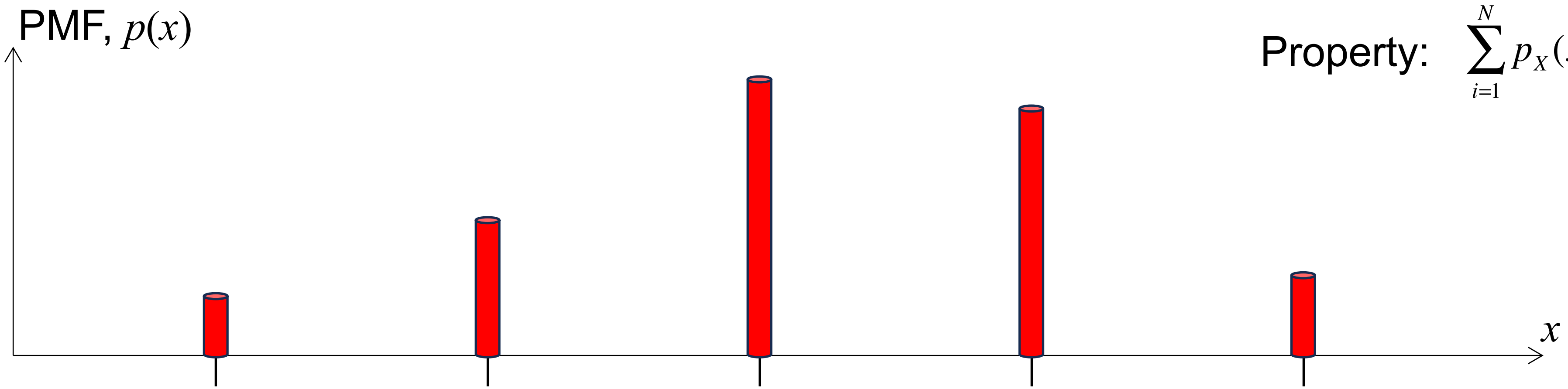


Continuous:



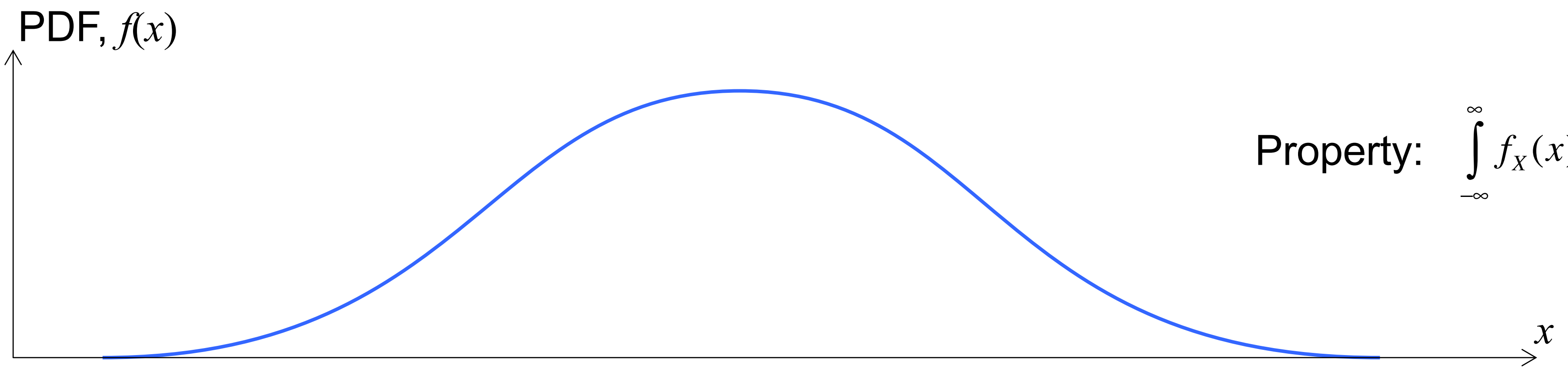
Probability Mass or Density

Discrete:



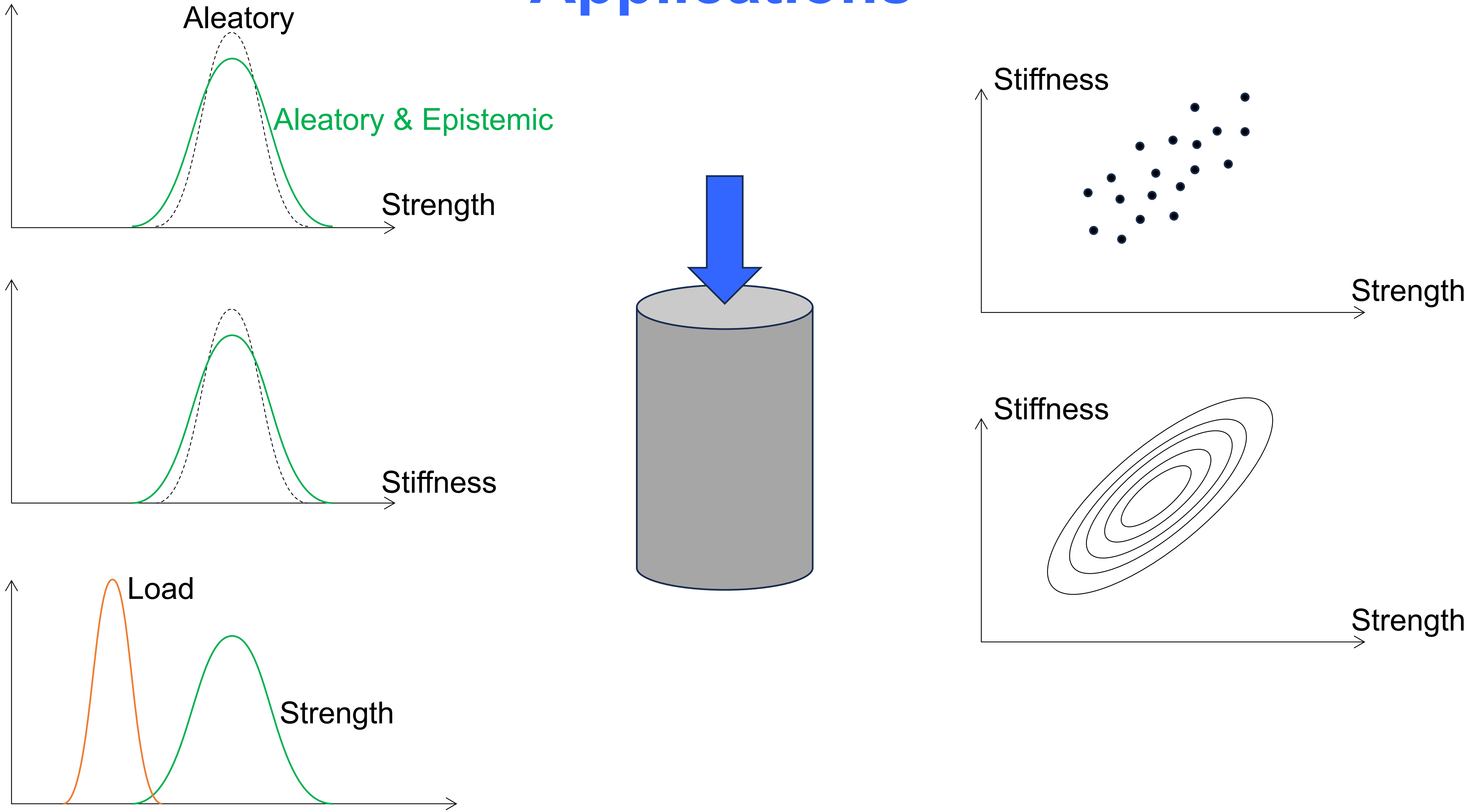
Property: $\sum_{i=1}^{N} p_X(x_i) = 1$

Continuous:



Property: $\int_{-\infty}^{\infty} f_X(x)\,dx = 1$

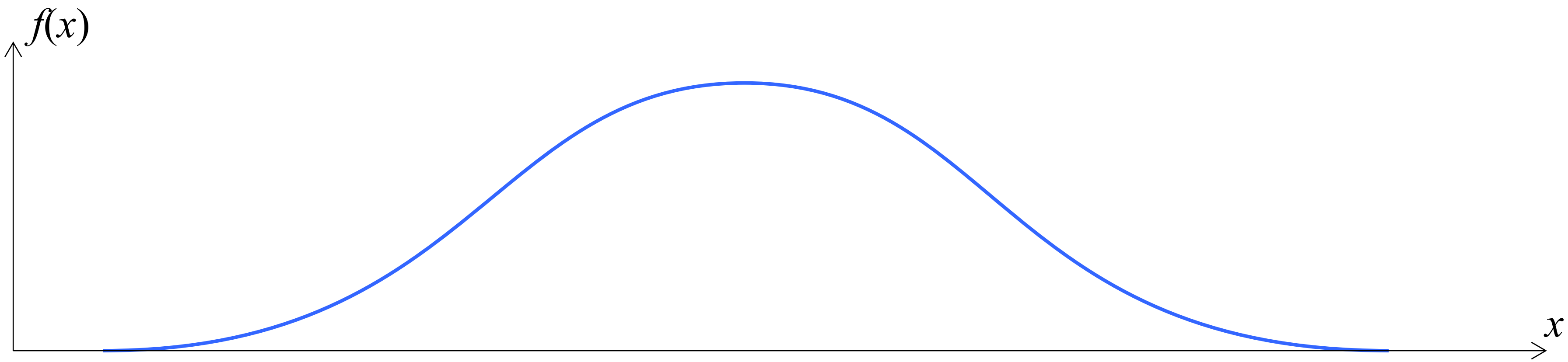
Applications



PDF is Density

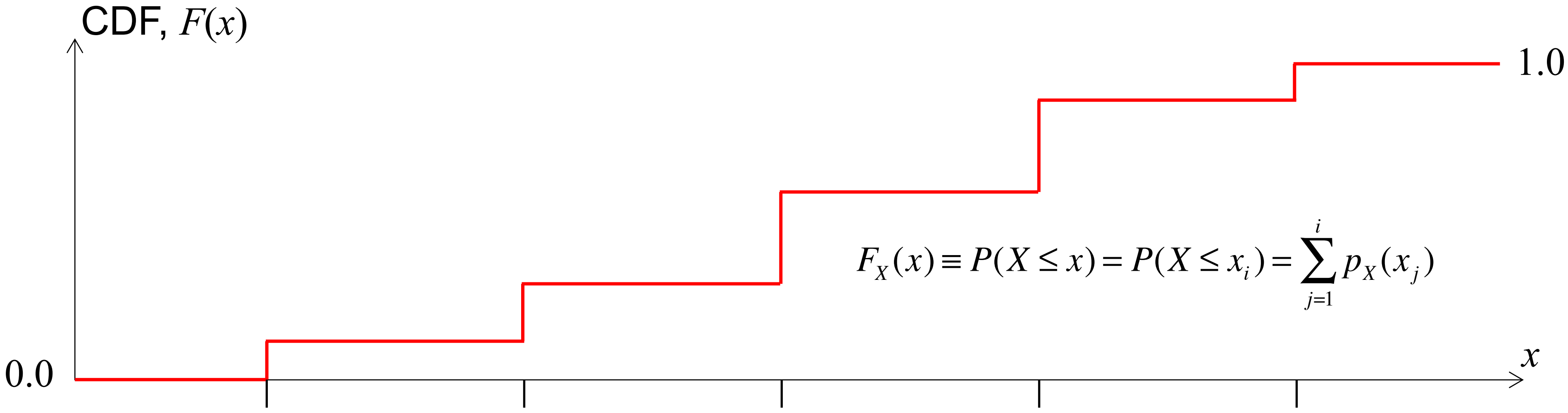
$$\mathrm{P}(X = x) = 0$$

$$f_X(x) \equiv P(x \leq X \leq x + dx)$$



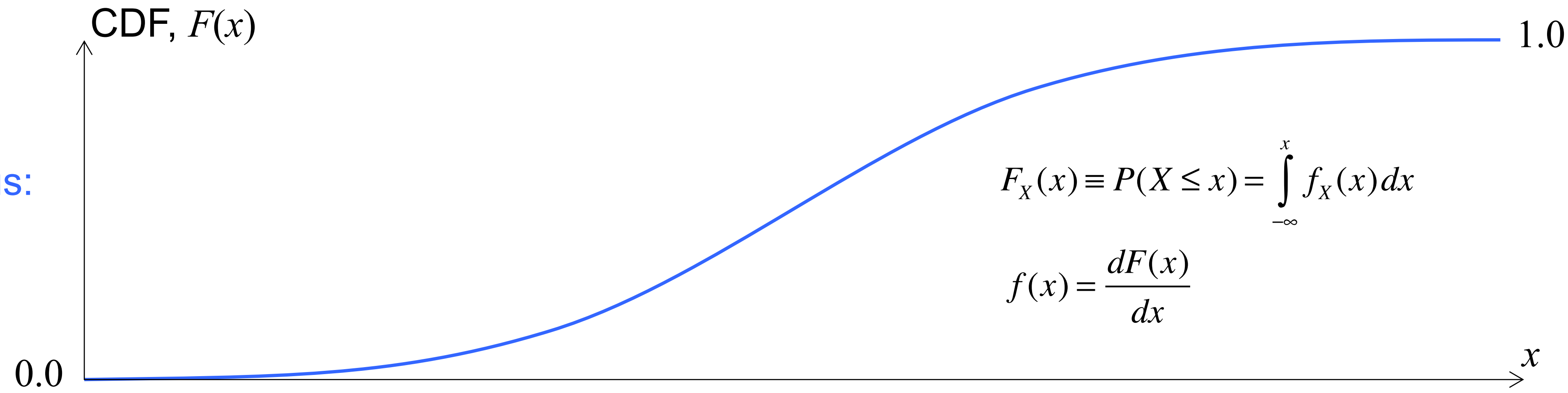
Cumulative Distribution Function

Discrete:



$$F_X(x) \equiv P(X \le x) = P(X \le x_i) = \sum_{j=1}^{i} p_X(x_j)$$

Continuous:



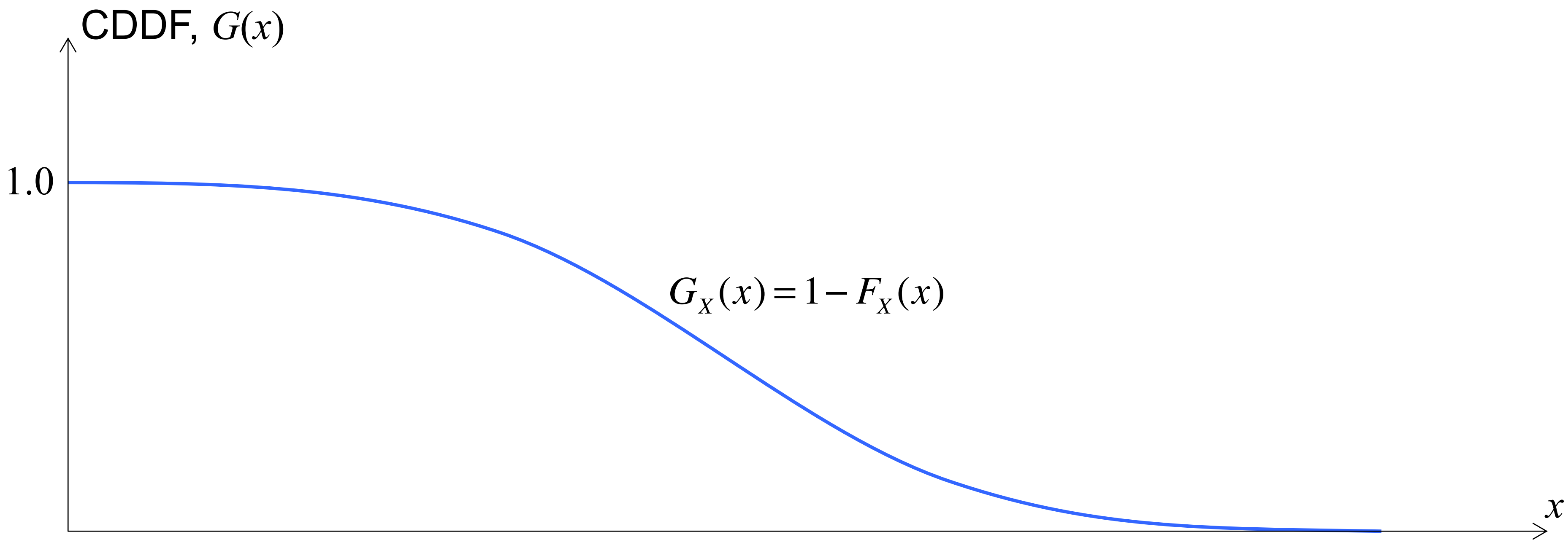
$$F_X(x) \equiv P(X \le x) = \int_{-\infty}^{x} f_X(x)\,dx$$

$$f(x) = \frac{dF(x)}{dx}$$

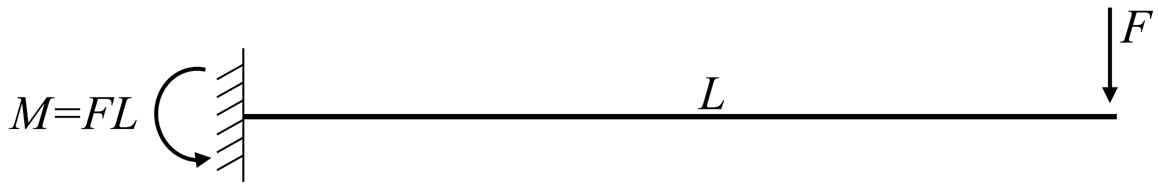
Complementary CDF

Discrete random variables: PMF → Sum gives CDF → Complement gives CCDF

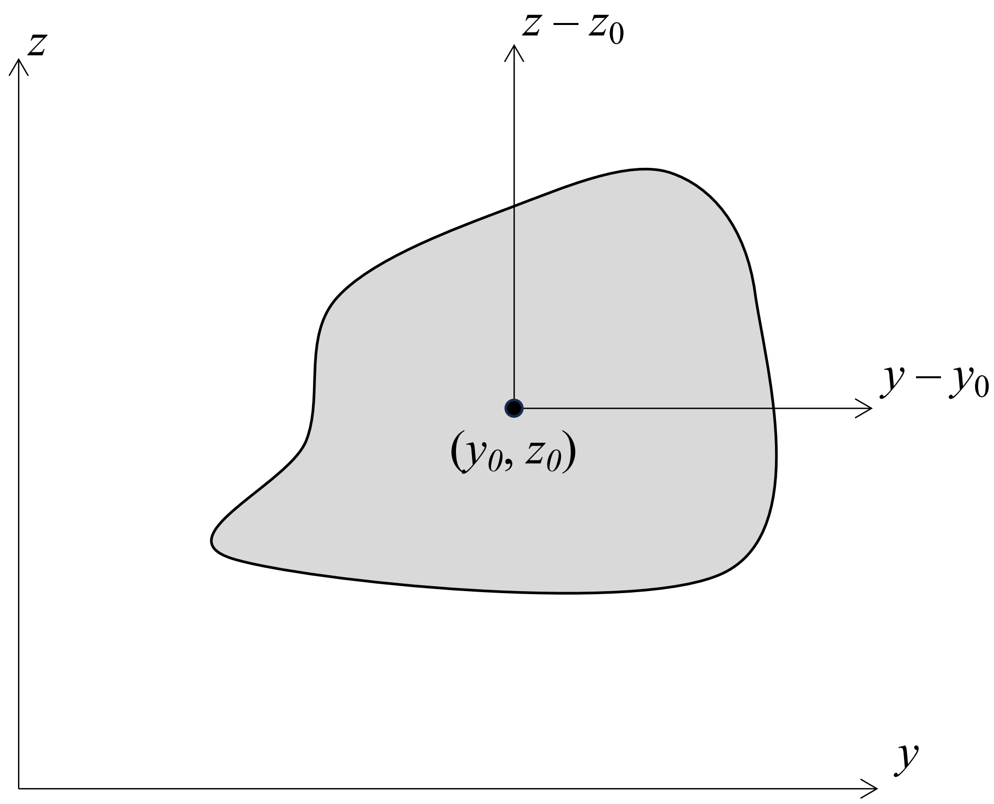
Continuous random variables: PDF → Integration gives CDF → Complement gives CCDF



Structural Moments

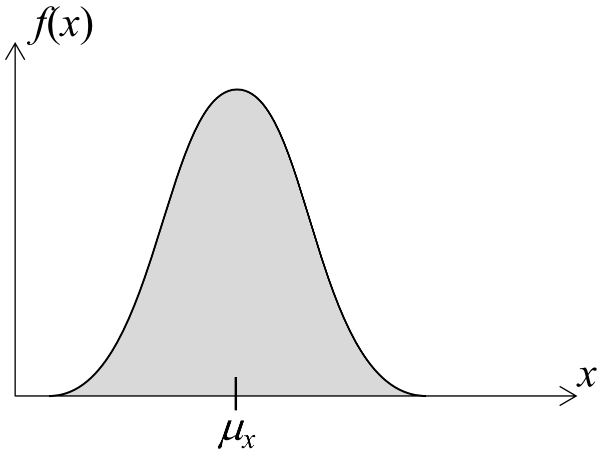


Cross-section Moments



	Moments	Central moments
First-order	$\int_A z \, dA$	$\int_A (z - z_0) \, dA = 0$
Second-order	$\int_A z^2 \, dA$	$I = \int_A (z - z_0)^2 \, dA$

Statistical Moments



	Moments	Central moments
First-order	$\mu_X = E[X] = \int_{-\infty}^{\infty} x \cdot f_X(x)\ dx$ (Mean)	$\int_{-\infty}^{\infty} (x - \mu_X) \cdot f_X(x)\ dx = 0$
Second-order	$E[X^2] = \int_{-\infty}^{\infty} x^2 \cdot f_X(x)\ dx$ (Mean square)	$\sigma_X^2 = \mathrm{Var}[X] = E[(x-\mu_X)^2] = \int_{-\infty}^{\infty} (x-\mu_X)^2 \cdot f_X(x)\ dx$ (Variance)

Poem

Variance is the mean square minus the square of the mean

$$\mathrm{Var}[X] = \int_{-\infty}^{\infty} (x-\mu)^2 \cdot f(x)\, dx = \int_{-\infty}^{\infty} x^2 \cdot f(x)\, dx + \int_{-\infty}^{\infty} \mu^2 \cdot f(x)\, dx - \int_{-\infty}^{\infty} 2 \cdot x \cdot \mu \cdot f(x)\, dx = E[X^2] - \mu^2$$

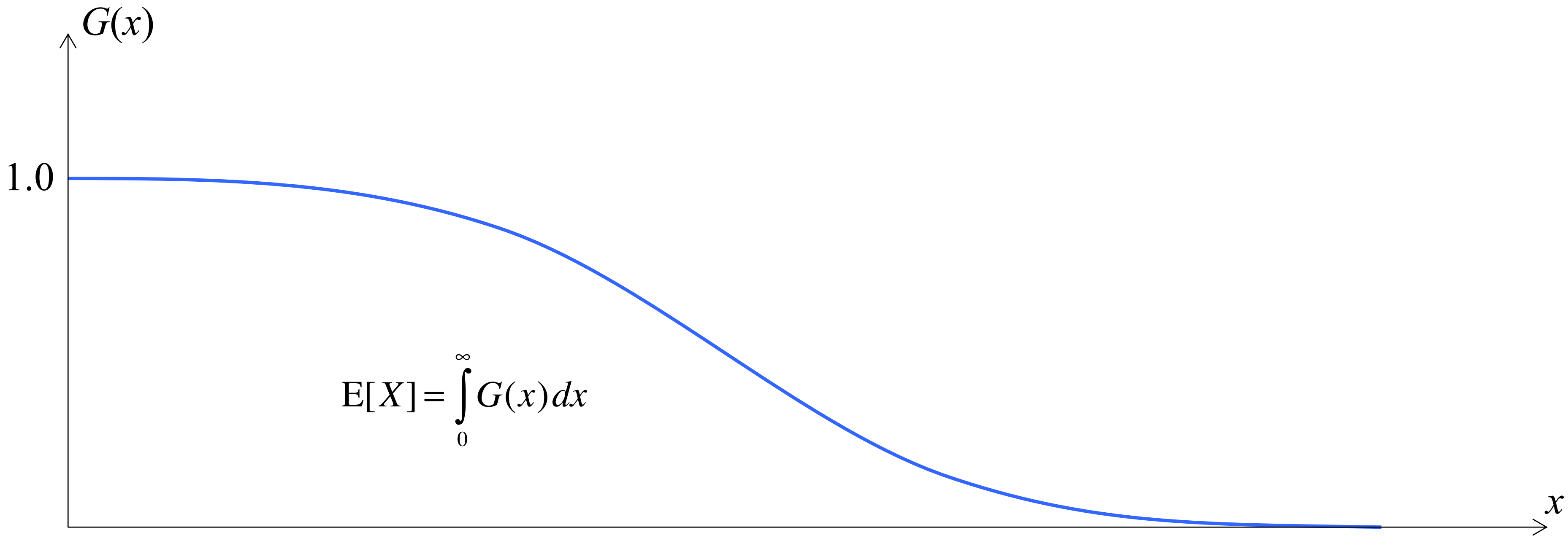
$$E[X^2]$$

$$\mu^2 \cdot \int_{-\infty}^{\infty} f(x)\, dx = \mu^2$$

$$2 \cdot \mu \cdot \int_{-\infty}^{\infty} x \cdot f(x)\, dx = 2 \cdot \mu^2$$

Mean from CCDF

$$\mu_X = E[X] = \int_{-\infty}^{\infty} x \cdot f_X(x)\ dx \ = -\int_{0}^{\infty} x \cdot \frac{G(x)}{dx} dx \ = -\left[x \cdot G(x)\right]_0^{\infty} + \int_0^{\infty} 1 \cdot G(x)\, dx$$



Coefficient of Variation

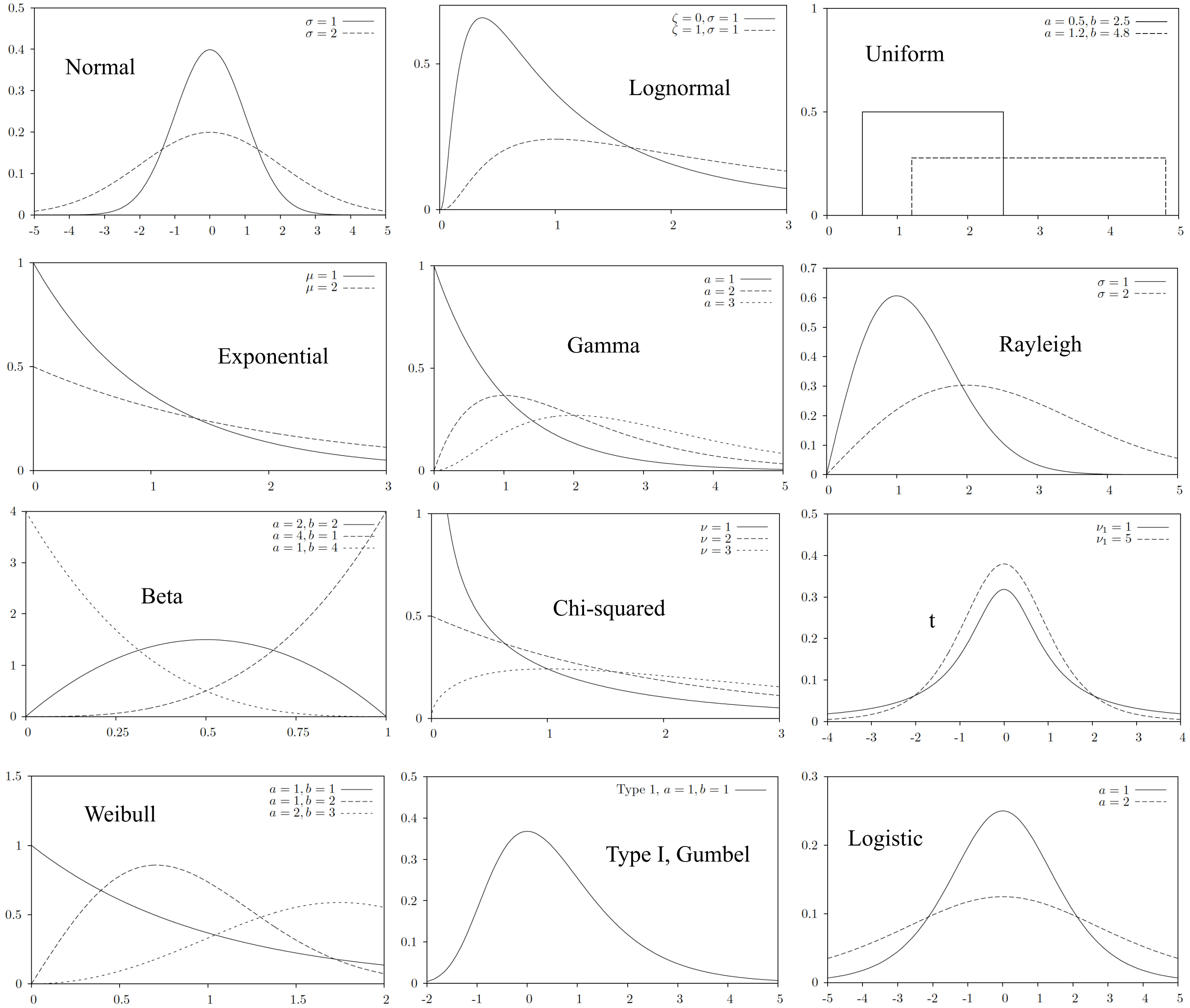
$$\delta_X = \frac{\sigma_X}{|\mu_X|}$$

Assuming 10% coefficient of variation, what is the standard deviation of the modulus of elasticity, *E*=210,000MPa?

$$\gamma_X = \frac{E[(x-\mu_X)^3]}{\sigma_X^3}$$

$$\kappa_X = \frac{E[(x-\mu_X)^4]}{\sigma_X^4}$$

Distribution Types

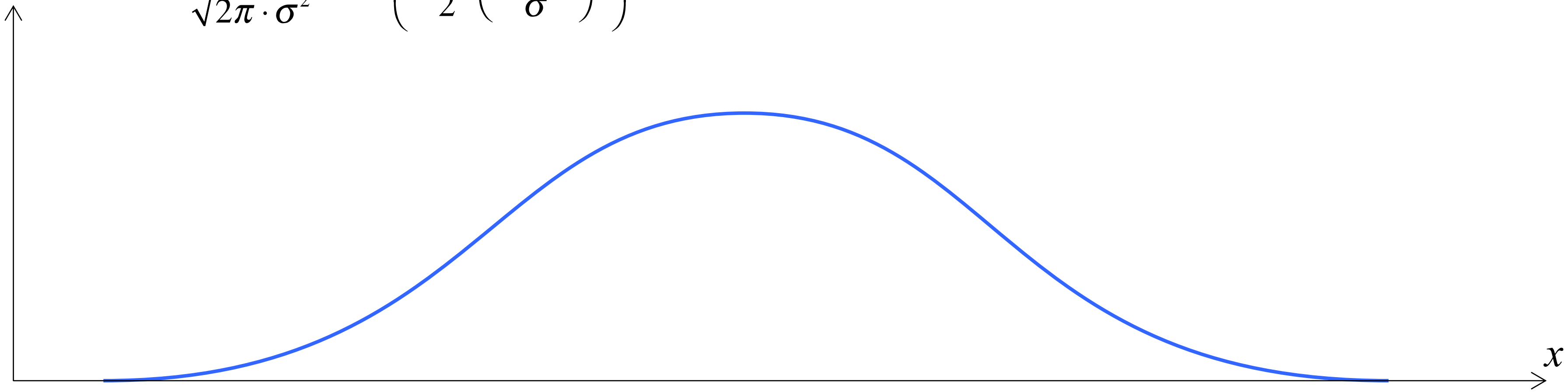


Normal

Central limit theorem

Sum of random phenomena

$$f(x,\mu,\sigma)=\frac{1}{\sqrt{2\pi\cdot\sigma^2}}\cdot\exp\left(-\frac{1}{2}\cdot\left(\frac{x-\mu}{\sigma}\right)^2\right)$$



Standard Normal

Zero mean, unit variance

PDF: $$\varphi(y) = \frac{1}{\sqrt{2\pi}} \cdot \exp\left(-\frac{y^2}{2}\right)$$ $$f(x) = \varphi\left(\frac{x-\mu}{\sigma}\right)$$

CDF: $$\Phi(y) = \int_{-\infty}^{y} \frac{1}{\sqrt{2\pi}} \cdot \exp\left(-\frac{y^2}{2}\right) dy$$ $$F(x) = \Phi\left(\frac{x-\mu}{\sigma}\right)$$

Lognormal

Product of random phenomena

$$X = Z_1 \cdot Z_2 \cdots Z_n$$

Central limit theorem

$$\ln(X) = \ln(Z_1) + \ln(Z_2) + \cdots + \ln(Z_n)$$

Definition of lognormal variable, X

$$Y = \ln(X)$$

Lognormal Distribution

$$Y = \ln(X)$$

Normal → Y ; Lognormal → X

Probability transformations:

$$F_X(x) = F_Y(y) = F_Y(\ln(x)) \quad \Rightarrow \quad F(x) = \Phi\left(\frac{\ln(x) - \mu_Y}{\sigma_Y}\right)$$

$$f_X(x) \cdot dx = f_Y(y) \cdot dy \quad \Rightarrow \quad f_X(x) = \frac{dy}{dx} \cdot f_Y(y) = \frac{1}{x} \cdot f_Y\left(\ln(x)\right) = \frac{1}{x} \cdot \varphi\left(\frac{\ln(x) - \mu_Y}{\sigma_Y}\right)$$

Notation

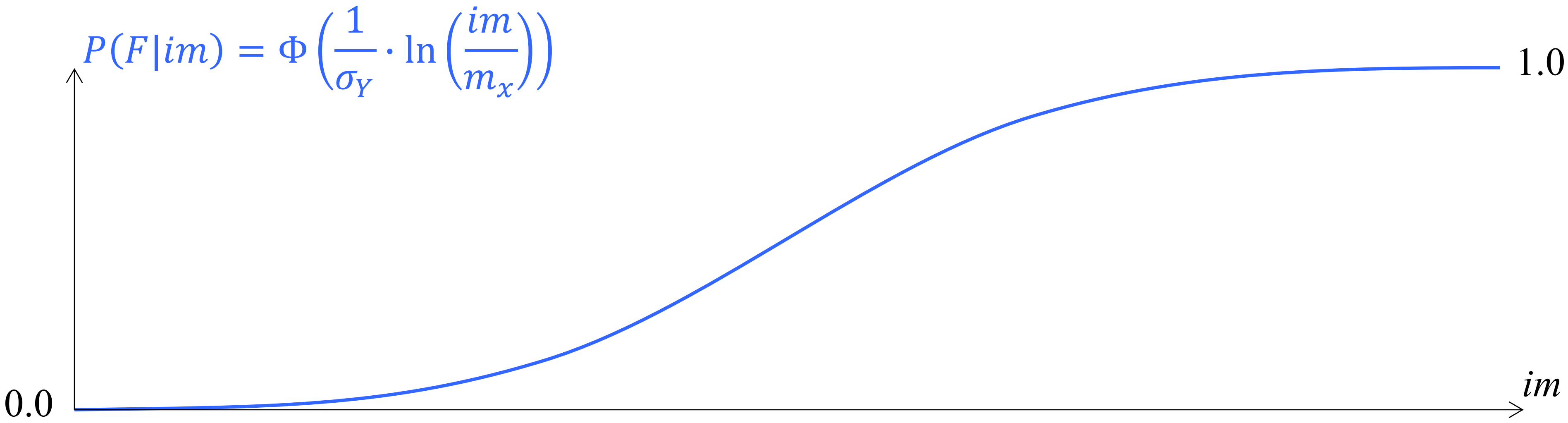
$$\mu_Y = \zeta$$
$$\sigma_Y = \sigma$$

PDF	$f(x,\zeta,\sigma) = \frac{1}{x \cdot \sqrt{2\pi \cdot \sigma^2}} \cdot \exp\left(-\frac{1}{2} \cdot \left(\frac{\ln(x) - \zeta}{\sigma}\right)^2\right)$
$mean = \mu_X =$	$\exp\left(\zeta + \frac{\sigma^2}{2}\right)$
$\text{stdv} = \sigma_X =$	$\sqrt{\exp(\sigma^2) - 1} \cdot \exp\left(\zeta + \frac{\sigma^2}{2}\right)$
$\zeta = \mu_Y = \ln(m_X) =$	$\ln(mean) - \frac{1}{2} \cdot \ln\left(1 + \left(\frac{stdv}{mean}\right)^2\right)$
$\sigma = \sigma_Y = \text{dispersion} =$	$\sqrt{\ln\left(\left(\frac{stdv}{mean}\right)^2 + 1\right)}$

Fragility Functions

$$\mu_Y = \ln(m_X)$$

$$\ln(x) - \mu_Y = \ln(x) - \ln(m_X) = \ln\left(\frac{x}{m_X}\right)$$



More lectures:

Terje’s Toobox:

terje.civil.ubc.ca