A short course on

Probabilities and Random Variables

This video:

Rules of Probability

Terje's Toolbox is freely available at terje.civil.ubc.ca
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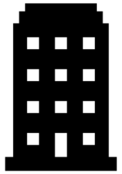
What do we use probabilities for?

- Will your bridge/building fail due to extreme loading in the next 50 years?

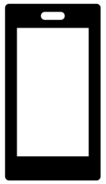


- Engineering = Decision-making under uncertainty
- Engineering = Design + Analysis
- Analysis = Models + Methods

Models: Random variables, Random functions, Stochastic processes, Etc.

Methods: Probabilistic methods, Reliability methods

- Compare result with target probability



- Compare risks: Risk = Cost * Probability

How do we express a probability?

One over a thousand?

One in a thousand?

A thousand to one?

0.1 percent chance?

999 to one?

$$P(F) = p_f = 10^{-3}$$

Reliability index of 3?

4995 to five?

Reliability index of 3.09?

It is a number between zero and one…

Odds & Reliability Index

The odds are "n to m"

$$P(F) = p_f = \Phi(-\beta)$$

$$1 - P(\text{event}) = \frac{n}{n+m}$$

$$\beta = -\Phi^{-1}(p_f)$$

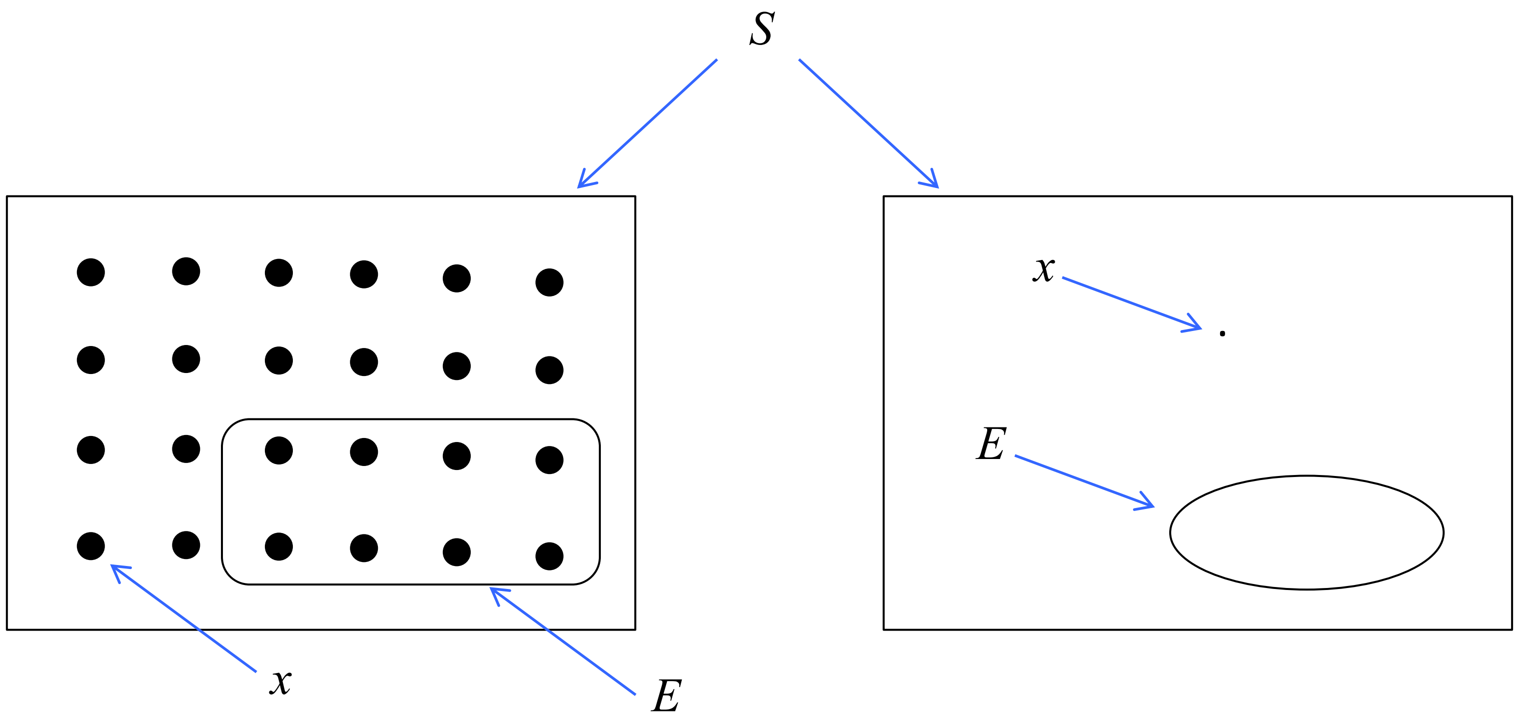
$$n = \frac{1 - P(\text{event})}{P(\text{event})}$$

What does a probability mean?

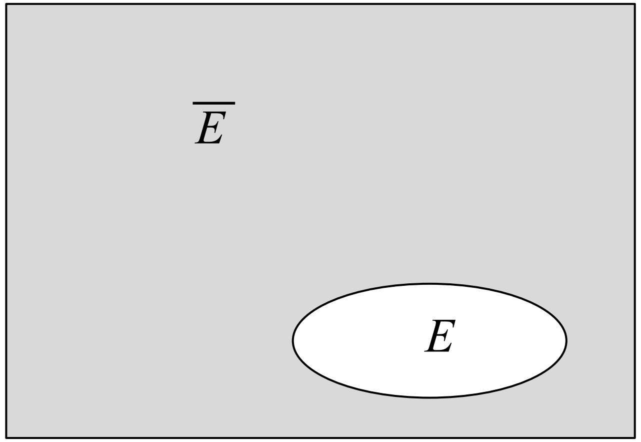
Frequentist, classical, $\mathrm{P}(E) = \lim_{n \to \infty} \frac{n_E}{n}$

Bayesian, subjectivist, degree of belief

Events & Venn Diagrams



Complement

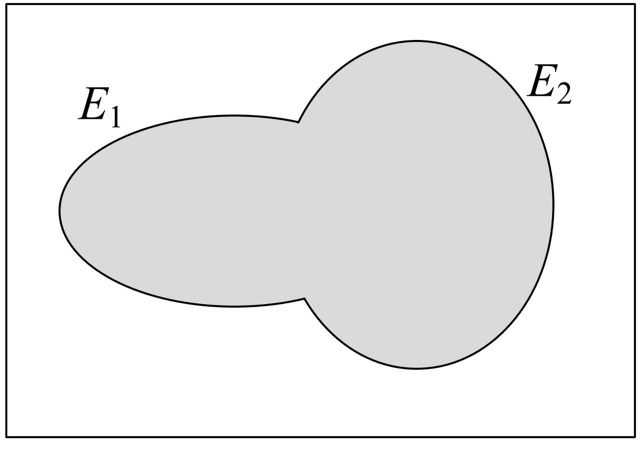


Ø is the complement of the **certain event**, S

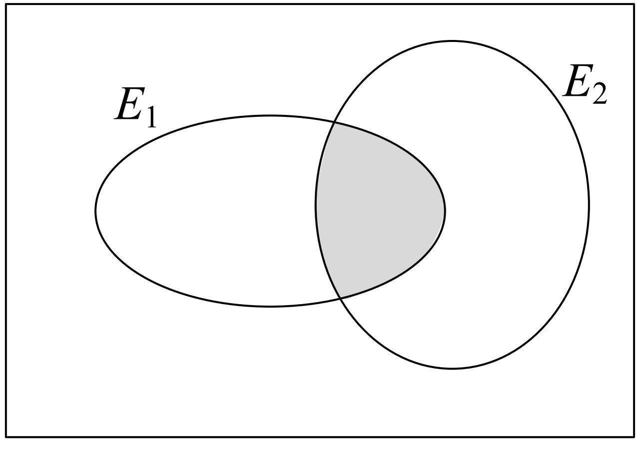
S is the complement of the **null event**, Ø

Union & Intersection

$E_1 \bigcup E_2$



$E_1 \bigcap E_2 \equiv E_1E_2$



Operator Rules

Commutative:

$$E_1 \cup E_2 = E_2 \cup E_1$$

$$E_1 \cap E_2 = E_2 \cap E_1 \equiv E_1 E_2$$

Associative:

$$\left(E_1 \cup E_2\right) \cup E_3 = E_1 \cup \left(E_2 \cup E_3\right)$$

$$\left(E_1 \cap E_2\right) \cap E_3 = E_1 \cap \left(E_2 \cap E_3\right) \equiv E_1 E_2 E_3$$

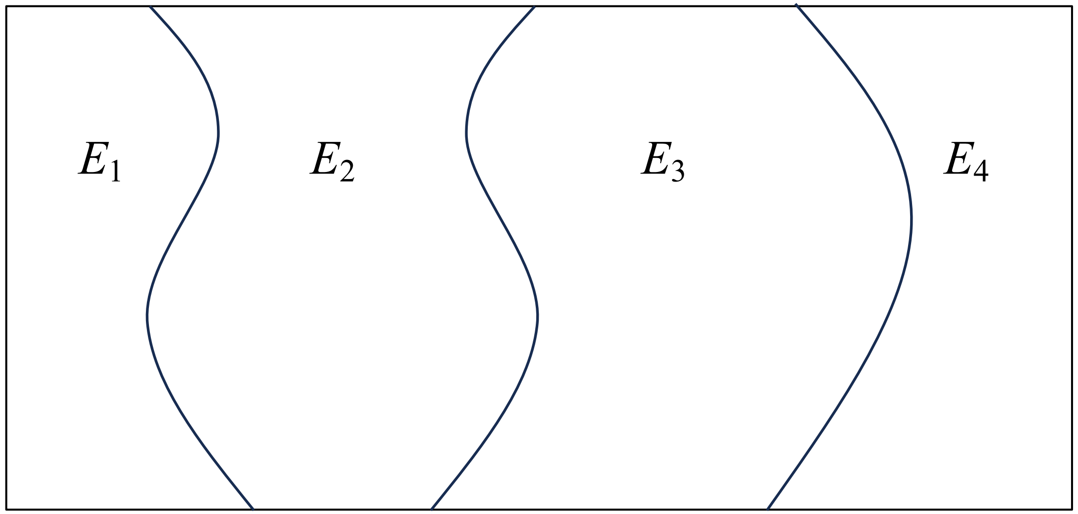
Distributive:

$$\left(E_1 \cup E_2\right) \cap E_3 = \left(E_1 \cap E_3\right) \cup \left(E_2 \cap E_3\right)$$

$$\left(E_1 \cap E_2\right) \cup E_3 = \left(E_1 \cup E_3\right) \cap \left(E_2 \cup E_3\right)$$

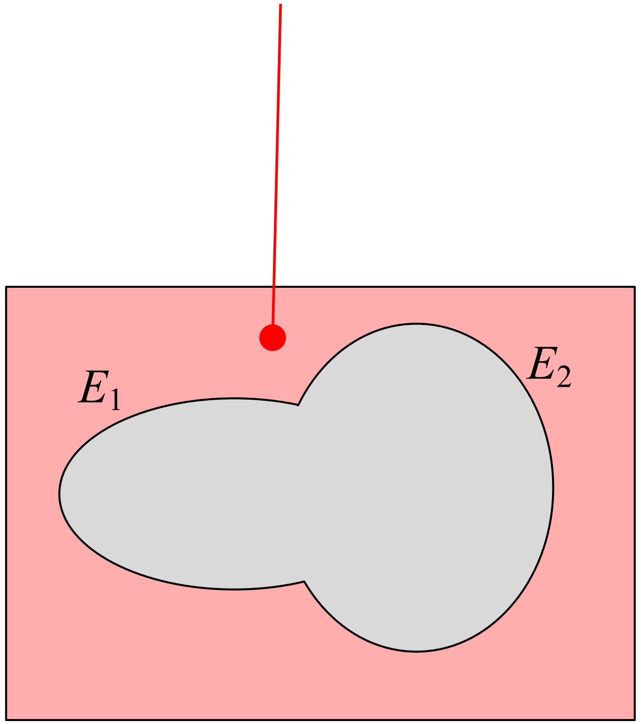
MECE

Mutually exclusive and collectively exhaustive

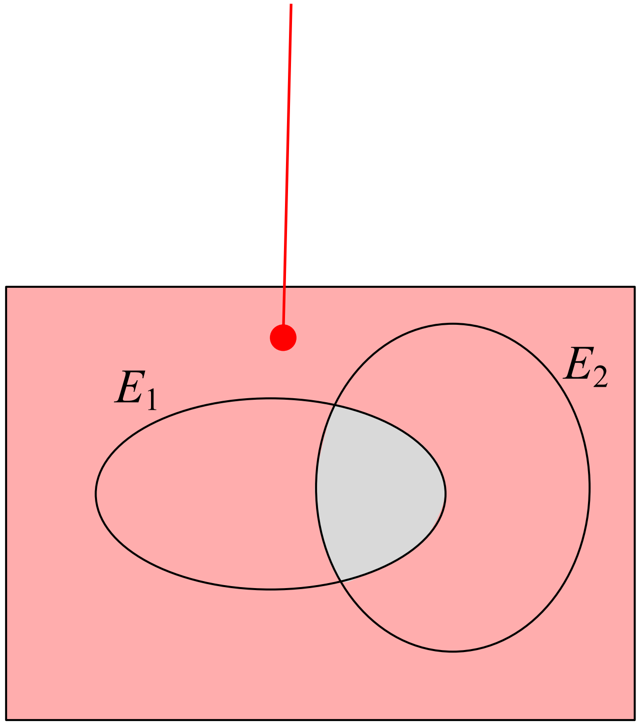


de Morgan's Rules

$$\overline{E_1 \cup E_2} = \bar{E}_1 \cap \bar{E}_2$$



$$\overline{E_1 \cap E_2} = \bar{E}_1 \cup \bar{E}_2$$



Axioms of Probability

Blaise Pascal & Pierre de Fermat (1654)

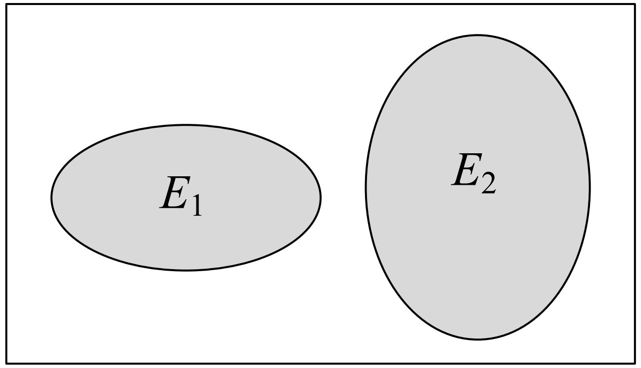
Pierre-Simon Laplace (1812)

Andrey Kolmogorov (1933)

$$P(E) \geq 0$$

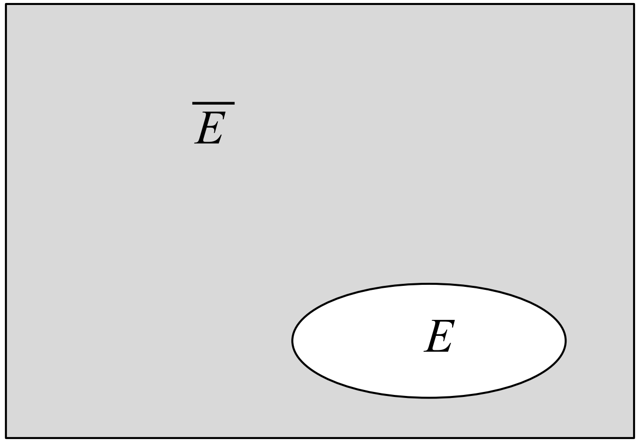
$$P(S) = 1$$

$$P(E_1 \cup E_2) = P(E_1) + P(E_2)$$



Probability of the Complement

$$P(\overline{E}) = 1 - P(E)$$



$$P(S) = P(E \cup \overline{E}) = P(E) + P(\overline{E}) = 1 \quad \Rightarrow \quad P(\overline{E}) = 1 - P(E)$$

Example

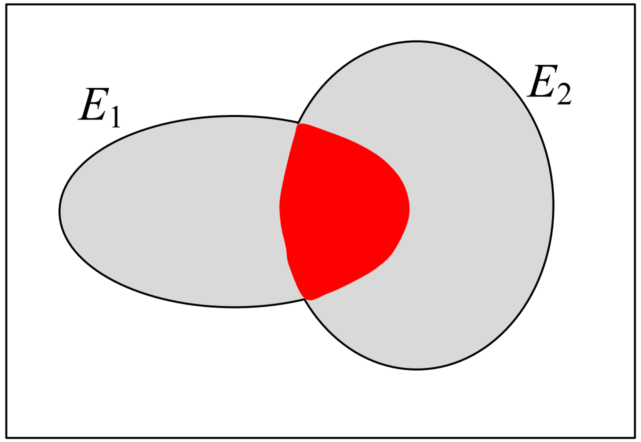
Probability of safe operations, when the failure probability is known

$$P(\text{safe}) = 1 - p_f$$

Union Rule

$$P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 E_2)$$

$E_1 \cup E_2$



Inclusion-Exclusion Rule

$$
\begin{aligned}
P(E_1 \cup E_2 \cup E_3) = & \; P(E_1) + P(E_2) + P(E_3) \\
& - P(E_1E_2) - P(E_1E_3) - P(E_2E_3) \\
& + P(E_1E_2E_3)
\end{aligned}
$$

Conditional Probability Rule

$$P(E_1 \mid E_2) = \frac{P(E_1 E_2)}{P(E_2)}$$

$$\frac{P(E_1 E_2)}{P(E_2)} = \frac{\left(\frac{n_{12}}{n}\right)}{\left(\frac{n_2}{n}\right)} = \frac{n_{12}}{n_2}$$

Multiplication Rule

Conditional probability rule

$$P(E_1 \mid E_2) = \frac{P(E_1 E_2)}{P(E_2)}$$

Multiplication rule

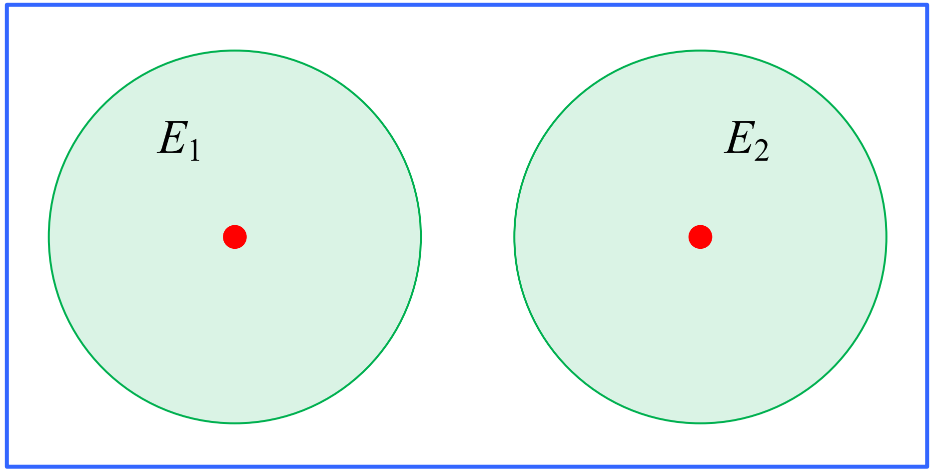
$$P(E_1 E_2) = P(E_1 \mid E_2) P(E_2)$$

Example

Bridge locations
Epicentre damage radius
Area source for earthquake occurrences

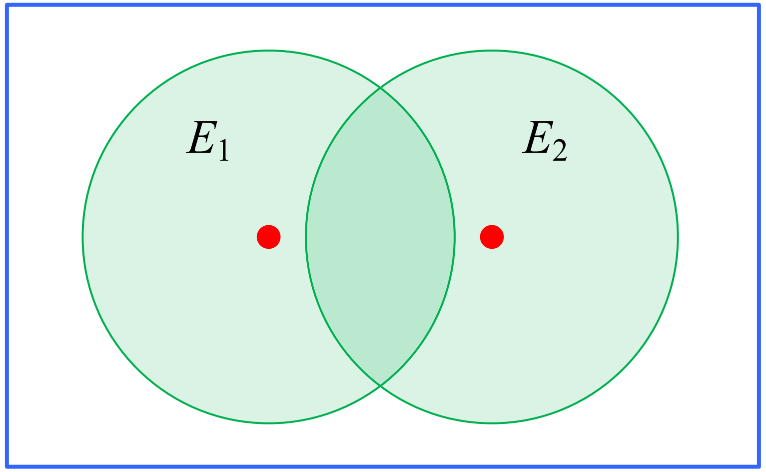
$$P(E_1) = P(E_2) = 0.1$$

$$P(E_1 \cup E_2) = ?$$



$$P(E_1 \cup E_2) = P(E_1) + P(E_2) = \underline{\mathbf{0.2}}$$

$$P(E_1 | E_2) = 0.5$$



$$P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1E_2)$$

$$= 0.1 + 0.1 - \mathbf{0.05} = \underline{\mathbf{0.15}}$$

$$P(E_1E_2) = P(E_1 | E_2)\, P(E_2) = (0.5)\,(0.1) = \mathbf{0.05}$$

Bayes' Rule

$$P(E_1 \mid E_2) = \frac{P(E_2 \mid E_1)}{P(E_2)} P(E_1)$$

Multiplication rule rule

$$P(E_1 \mid E_2) = \frac{P(E_1 E_2)}{P(E_2)} = \frac{P(E_2 \mid E_1) \cdot P(E_1)}{P(E_2)}$$

Conditional probability rule

Rule of Total Probability

$$P(A) = \sum_{i=1}^{n} P(A \mid E_i) P(E_i)$$

$$\begin{aligned}
P(A) &= P(AS) \\
&= P\left(A(E_1 \cup E_2 \cup \cdots \cup E_n)\right) \\
&= P(AE_1 \cup AE_2 \cup \cdots \cup AE_n) \\
&= \sum_{i=1}^{n} P(AE_i) \\
&= \sum_{i=1}^{n} P(A \mid E_i) P(E_i)
\end{aligned}$$

Example

F = flawed wood specimen

D = detection of flaw in a test

In general, P(F) = 0.008

The test device has P(D|F) = 0.9 and $P(D|\overline{F})$ = 0.07

What is the probability that a product that is identified as flawed in a test is actually flawed?

$$P(F|D) = \frac{P(D|F)}{P(D)}P(F) = \frac{0.9}{\mathbf{0.07664}} \cdot 0.008 = \mathbf{0.094}$$

$$P(D) = P(D|F)P(F) + P(D|\overline{F})P(\overline{F}) = (0.9)(0.008) + (0.07)(1 - 0.008) = \mathbf{0.07664}$$

Example

S = strong ground shaking
M = moderate ground shaking
W = weak ground shaking
F = failure of the facility

Moderate ground shaking is three times more likely than strong shaking and weak shaking is three times more likely than moderate shaking

P(F) = 0.3, 0.05 and 0.001 for S, M, and W shaking, respectively

What is P(F)?

$P(F) = P(F|S)P(S)+P(F|M)P(M)+P(F|W)P(W) = (0.3)\ (0.077) + (0.05)\ (0.231) + (0.001)\ (0.629) = \underline{\mathbf{0.035}}$

The rule of total probability requires MECE events, whose probability must add up to unity:

$P(S) = 1 / (1+3+9) = \mathbf{0.077}$ $P(M) = 3 / (1+3+9) = \mathbf{0.231}$ $P(W) = 9 / (1+3+9) = \mathbf{0.692}$

What is the probability that the earthquake was strong if the facility got damaged? Bayes' rule:

$P(S|F) = P(F|S)\ P(S) / P(F) = (0.3)\ (0.077) / 0.035 = \underline{\mathbf{0.65}}$

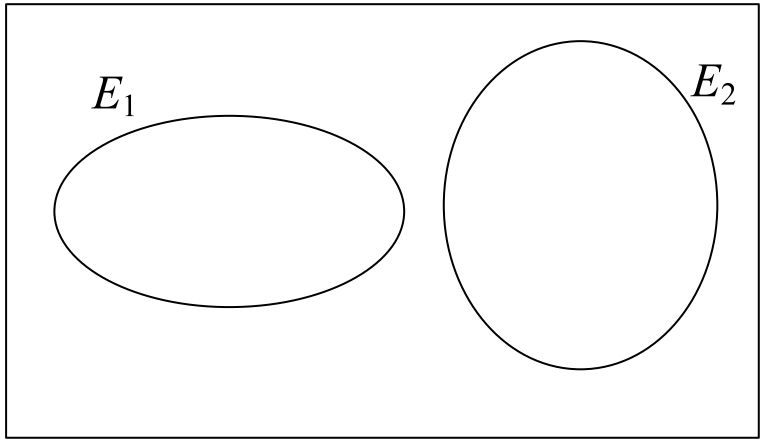
Statistical Dependence

Definition of INDEPENDENCE:

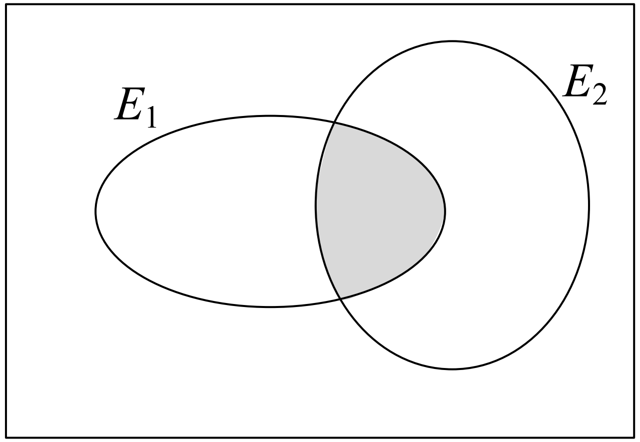
$$P(E_1 \mid E_2) = P(E_1)$$

Consequence of INDEPENDENCE:

$$P(E_1 E_2) = P(E_1) P(E_2)$$

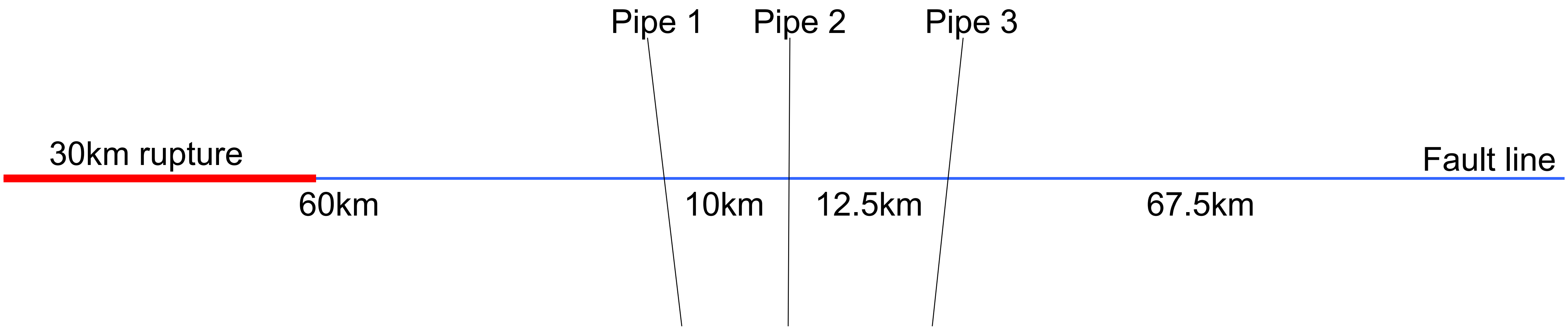


Dependent



Independent?

Example



Are the failure of the pipelines statistically independent?

$$P(F_1) = P(F_2) = P(F_3) = \frac{30\text{km}}{150\text{km} - 30\text{km}} = \mathbf{0.25}$$

$$P(F_1F_2) = \frac{30\text{km} - 10\text{km}}{150\text{km} - 30\text{km}} = 0.167$$

$$P(F_1|F_2) = \frac{0.167}{0.25} = 0.668$$

$$P(F_1F_3) = \frac{30\text{km} - 10\text{km} - 12.5\text{km}}{150\text{km} - 30\text{km}} = 0.0625$$

$$P(F_1|F_3) = \frac{0.0625}{0.25} = \mathbf{0.25}$$

$$P(F_2F_3) = \frac{30\text{km} - 12.5\text{km}}{150\text{km} - 30\text{km}} = 0.1458$$

$$P(F_2|F_3) = \frac{0.1458}{0.26} = 0.583$$

More lectures:

Terje's Toobox:

terje.civil.ubc.ca