A short course on

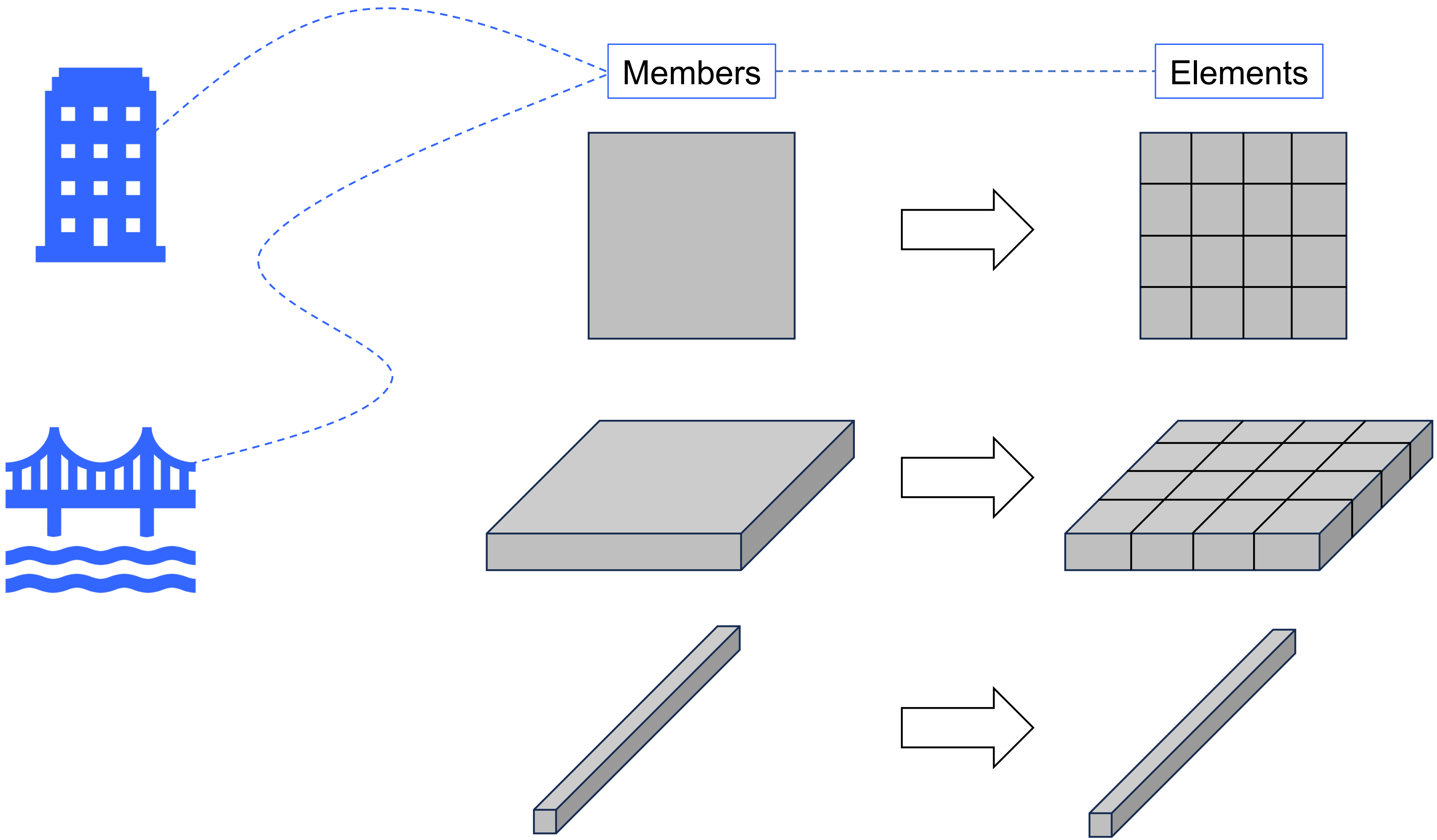
# The Finite Element Method

This video:

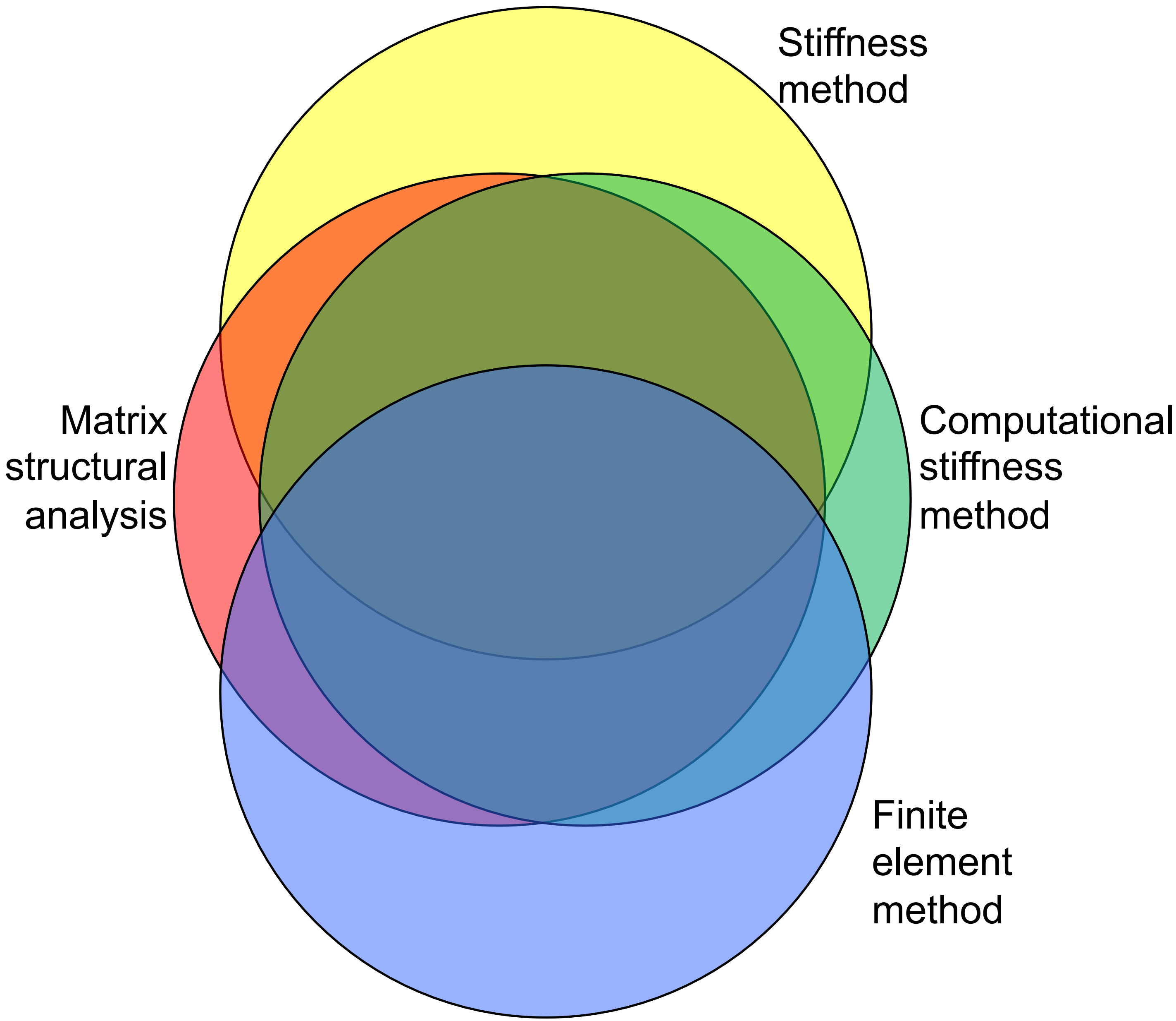
**Computational Stiffness Method & Matrix Structural Analysis**

Terje's Toolbox is freely available at terje.civil.ubc.ca
It is created and maintained by Professor Terje Haukaas, Ph.D., P.Eng.,
Department of Civil Engineering, The University of British Columbia (UBC), Vancouver, Canada

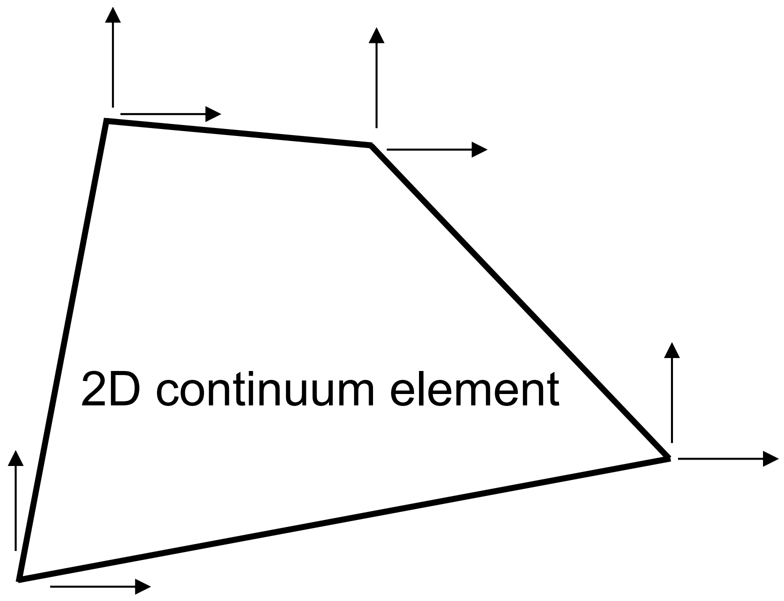
# Members & Elements

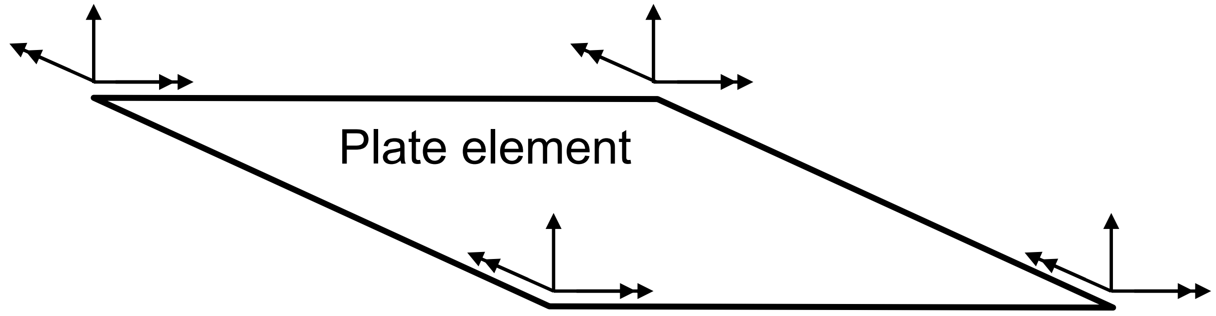


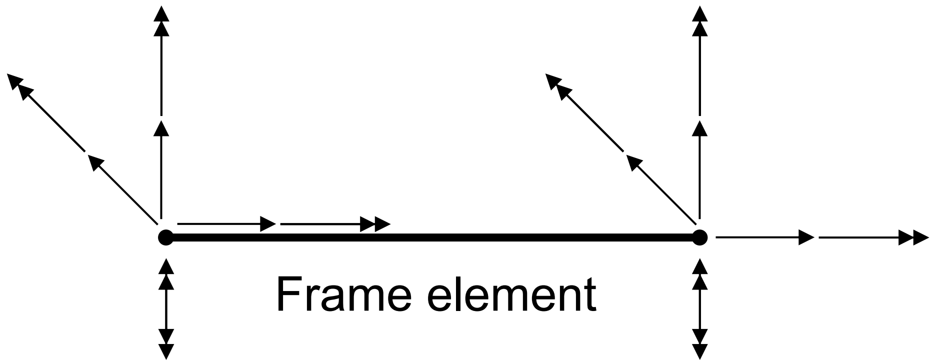
# Methods


Stiffness method
Matrix structural analysis
Computational stiffness method
Finite element method

# Elements & Degrees of Freedom



# Stiffness Method

$$K_{11}\,u_1 + K_{12}\,u_2 = F_1$$
$$K_{21}\,u_1 + K_{22}\,u_2 = F_2$$

$$\mathbf{Ku=F}$$

$$\delta W_{int} = \delta W_{ext}$$ (PVD)

# Finite Element Method

# Hand Calculations

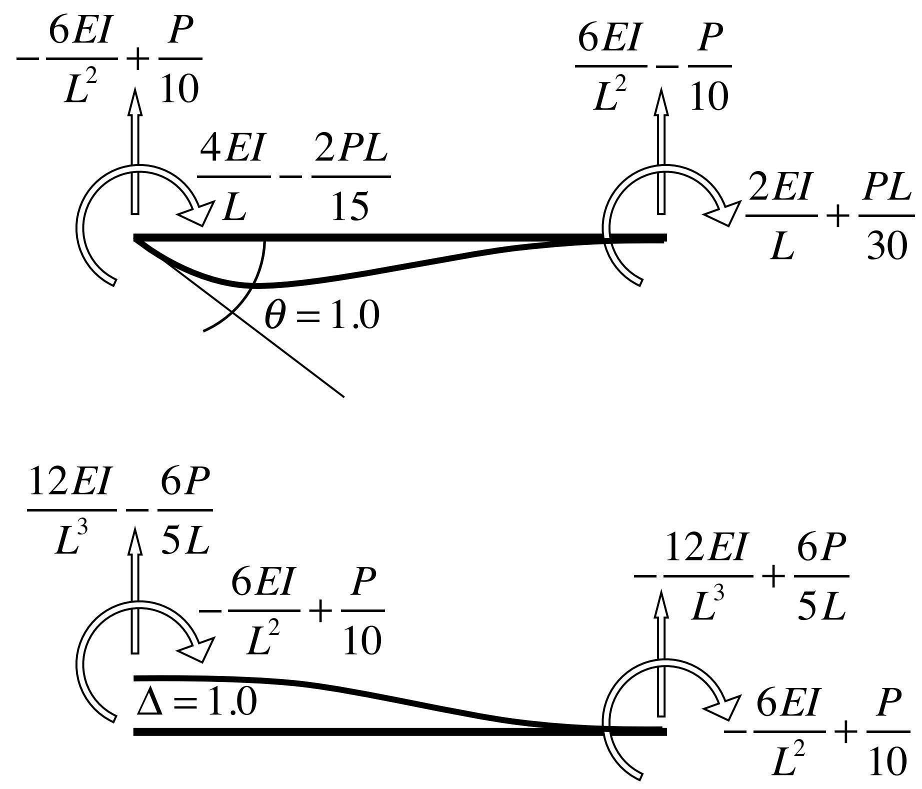
$K_{ij}$ = force along DOF $i$ due to unit displacement/rotation along DOF $j$

Sketch displaced shape for a unit displacement along DOF $j$, with all other DOFs clamped

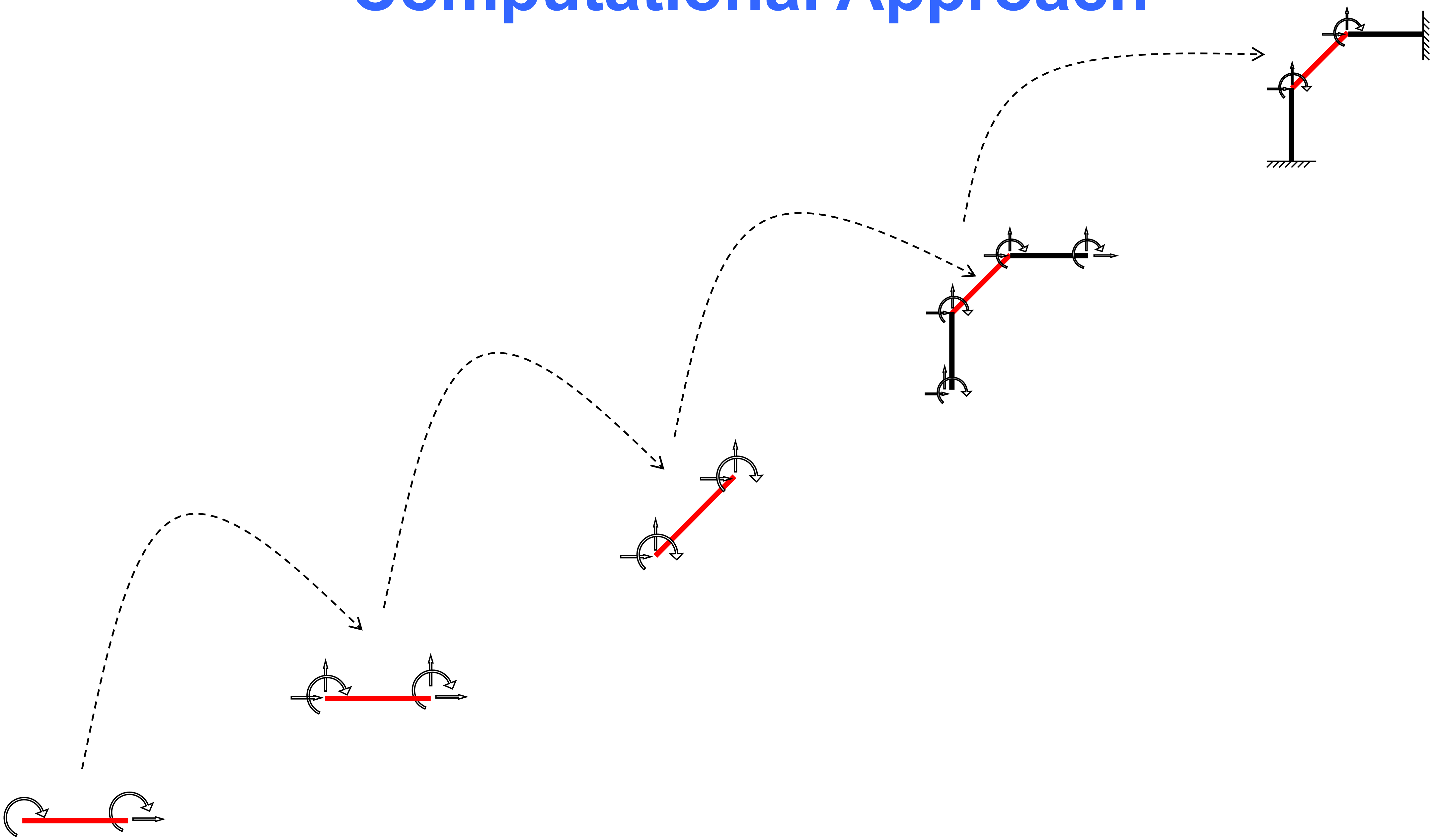
Determine force along DOFs to maintain that displaced shape, i.e., $K_{ij}$, which forms column number $j$ of $\mathbf{K}$

Do that for all DOFs to establish all columns of $\mathbf{K}$

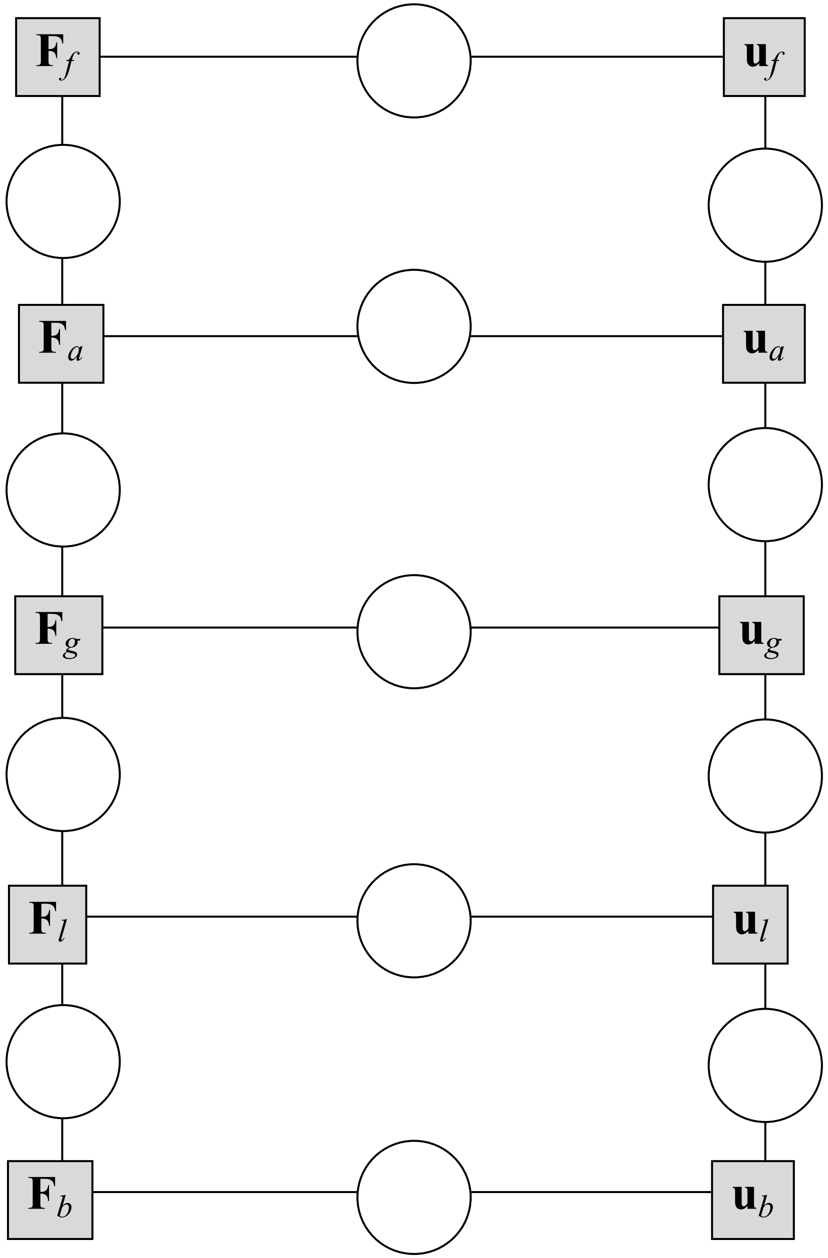
Check that $\mathbf{K}$ is symmetric with positive diagonal

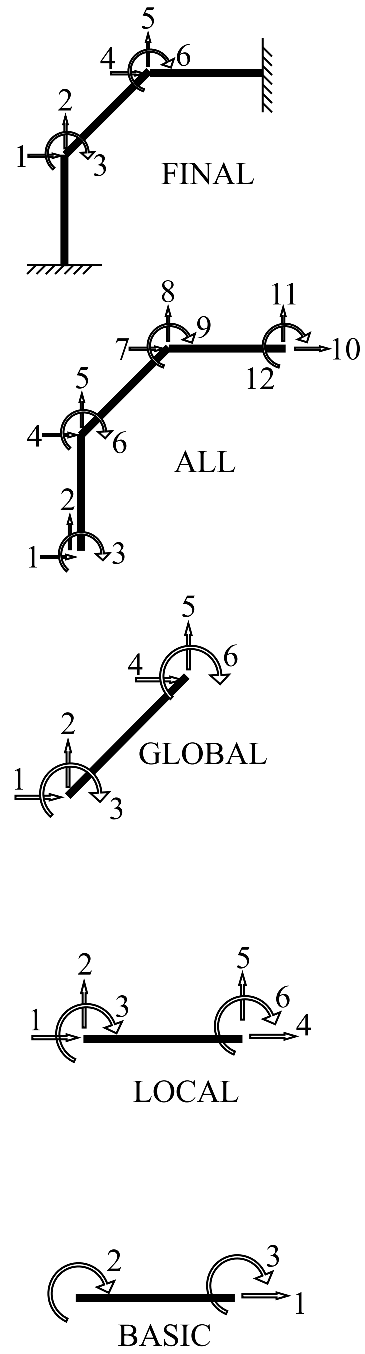


# Computational Approach



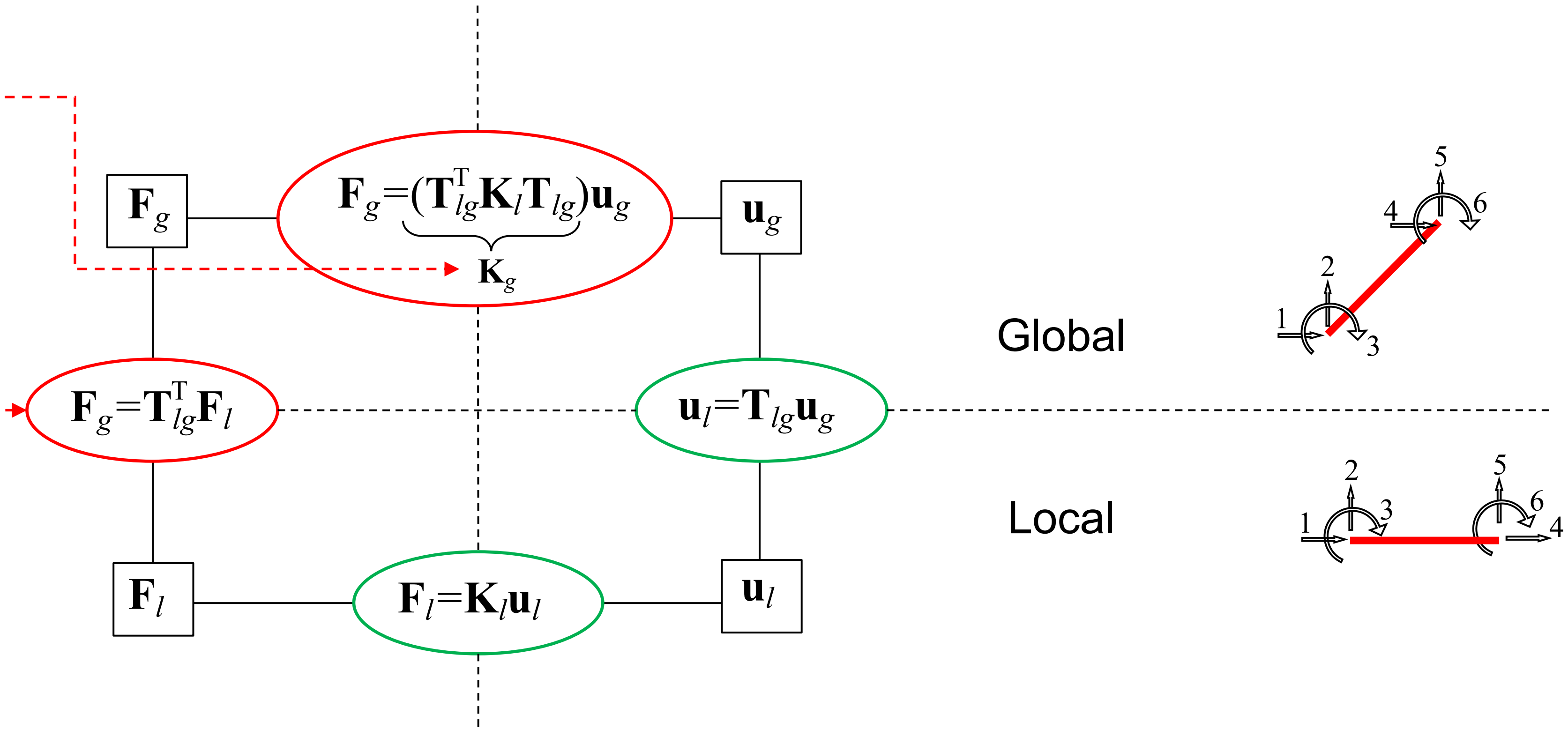
# Configurations



# Transformation Matrices

$$\mathbf{F}_g = \mathbf{T}_{lg}^{\mathrm{T}}\mathbf{F}_l = \mathbf{T}_{lg}^{\mathrm{T}}\mathbf{K}_l\mathbf{u}_l = \underbrace{\mathbf{T}_{lg}^{\mathrm{T}}\mathbf{K}_l\mathbf{T}_{lg}}_{\mathbf{K}_g}\mathbf{u}_g$$




$$\mathbf{F}_g^T\delta\mathbf{u}_g - \mathbf{F}_l^T\delta\mathbf{u}_l = 0$$

$$\mathbf{F}_g^T\delta\mathbf{u}_g - \mathbf{F}_l^T\mathbf{T}_{lg}\delta\mathbf{u}_g = \left(\mathbf{F}_g^T - \mathbf{F}_l^T\mathbf{T}_{lg}\right)\delta\mathbf{u}_g = 0$$

$$\mathbf{F}_g^T - \mathbf{F}_l^T\mathbf{T}_{lg} = 0 \quad\Rightarrow\quad \mathbf{F}_g - \mathbf{T}_{\mathrm{lg}}^T\mathbf{F}_l = 0 \quad\Rightarrow\quad \mathbf{F}_g = \mathbf{T}_{\mathrm{lg}}^T\mathbf{F}_l$$

# Summary

$$\mathbf{F}_f \;—\; \mathbf{K}_f=\mathbf{T}_{af}^{T}\mathbf{K}_a\mathbf{T}_{af} \;—\; \mathbf{u}_f$$

$$\mathbf{F}_f=\mathbf{T}_{af}^{T}\mathbf{F}_a \qquad \mathbf{u}_a=\mathbf{T}_{af}\mathbf{u}_f$$

$$\mathbf{F}_a \;—\; \mathbf{K}_a=\Sigma\left(\mathbf{T}_{ga}^{T}\mathbf{K}_g\mathbf{T}_{ga}\right) \;—\; \mathbf{u}_a$$

$$\mathbf{F}_a=\Sigma\left(\mathbf{T}_{ga}^{T}\mathbf{F}_g\right) \qquad \mathbf{u}_g=\mathbf{T}_{ga}\mathbf{u}_a$$

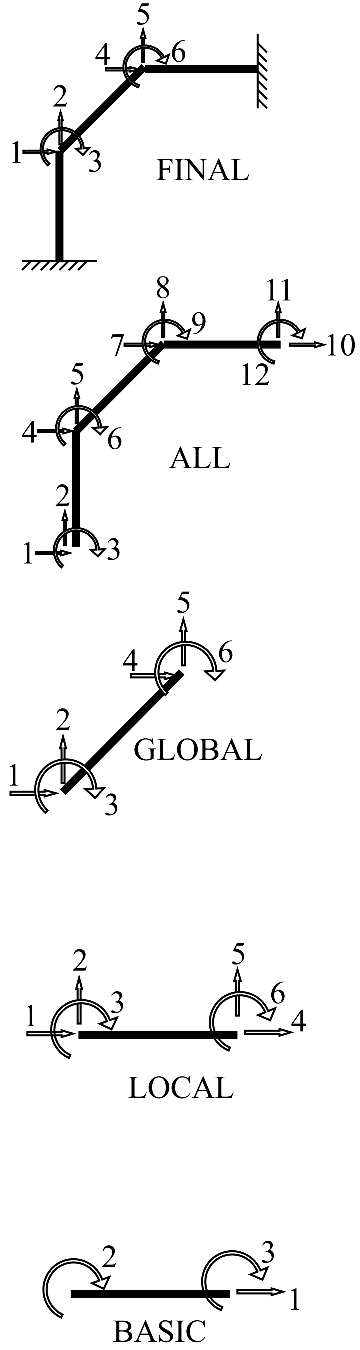
$$\mathbf{F}_g \;—\; \mathbf{K}_g=\mathbf{T}_{lg}^{T}\mathbf{K}_l\mathbf{T}_{lg} \;—\; \mathbf{u}_g$$

$$\mathbf{F}_g=\mathbf{T}_{lg}^{T}\mathbf{F}_l \qquad \mathbf{u}_l=\mathbf{T}_{lg}\mathbf{u}_g$$

$$\mathbf{F}_l \;—\; \mathbf{K}_l=\mathbf{T}_{bl}^{T}\mathbf{K}_b\mathbf{T}_{bl} \;—\; \mathbf{u}_l$$

$$\mathbf{F}_l=\mathbf{T}_{bl}^{T}\mathbf{F}_b \qquad \mathbf{u}_b=\mathbf{T}_{bl}\mathbf{u}_l$$

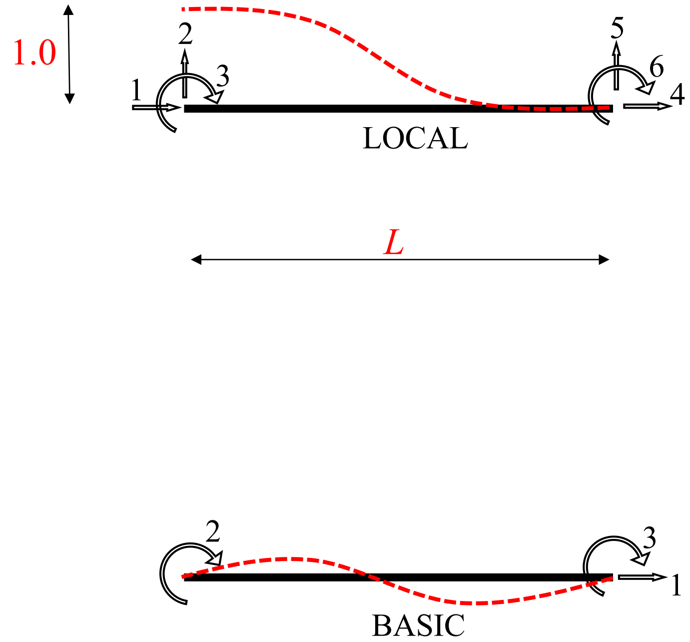
$$\mathbf{F}_b \;—\; \mathbf{K}_b \;—\; \mathbf{u}_b$$



# Basic → Local

$$\mathbf{u}_b = \mathbf{T}_{bl}\mathbf{u}_l$$

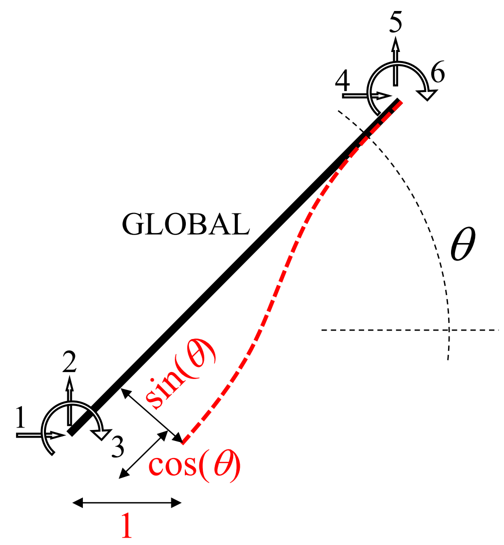
$$\mathbf{T}_{bl} = \left[\begin{array}{ccc:ccc} -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1/L & 1 & 0 & 1/L & 0 \\ 0 & -1/L & 0 & 0 & 1/L & 1 \end{array}\right]$$

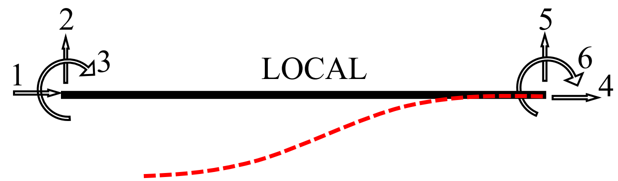


# Local → Global

$$\mathbf{u}_l = \mathbf{T}_{lg}\mathbf{u}_g$$

$$\mathbf{T}_{lg} = \left[\begin{array}{ccc|ccc} \cos(\theta) & \sin(\theta) & 0 & 0 & 0 & 0 \\ -\sin(\theta) & \cos(\theta) & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & \cos(\theta) & \sin(\theta) & 0 \\ 0 & 0 & 0 & -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

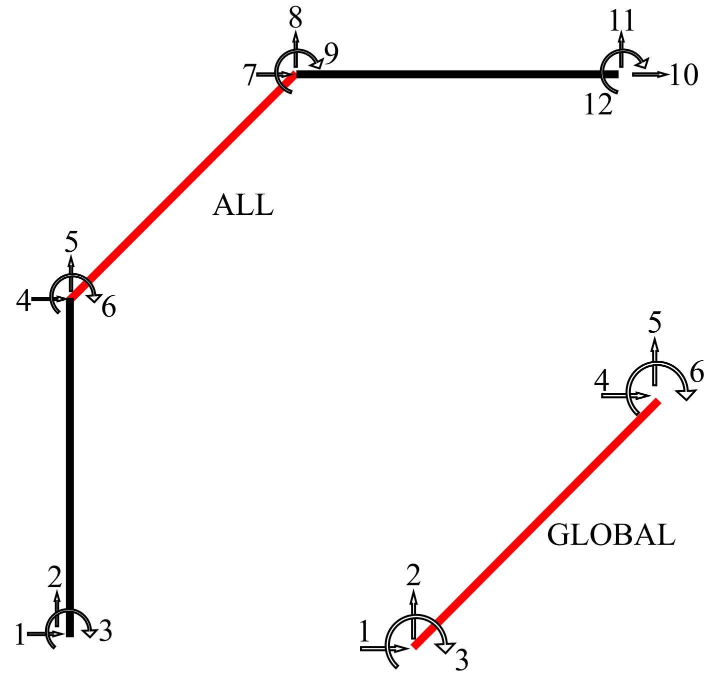


# Global → All

$$\mathbf{u}_g = \mathbf{T}_{ga}\mathbf{u}_a$$

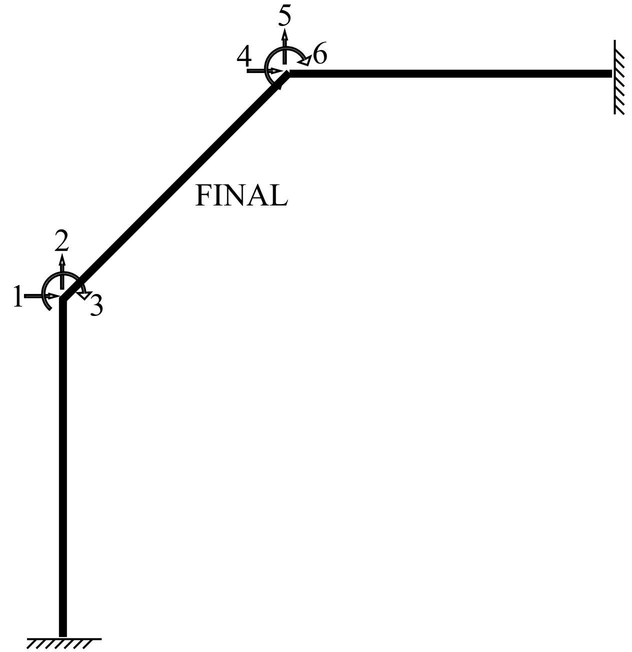
$$\mathbf{T}_{ga} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

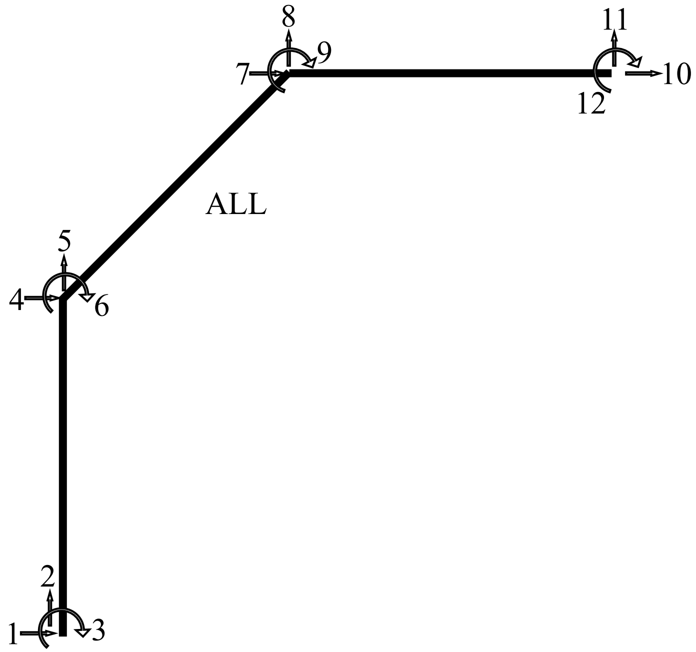


# All → Final

$$\mathbf{u}_a = \mathbf{T}_{af}\mathbf{u}_f$$

$$\mathbf{T}_{af} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$



# Inefficient Assembly

$$\mathbf{K}_f = \mathbf{T}_{af}^T \left( \sum_{i=1}^{numEl} \mathbf{T}_{ga,i}^T \mathbf{T}_{\mathrm{lg},i}^T \mathbf{T}_{bl,i}^T \mathbf{k}_{b,i} \mathbf{T}_{bl,i} \mathbf{T}_{\mathrm{lg},i} \mathbf{T}_{ga,i} \right) \mathbf{T}_{af}$$

# Efficient Assembly

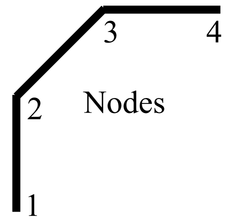
$$\mathbf{DOF} = \begin{bmatrix} node_1 \\ node_2 \\ node_3 \\ node_4 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{bmatrix}$$

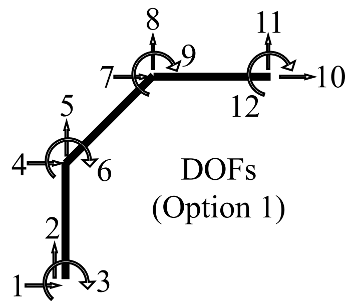
$$\mathbf{ID} = \begin{bmatrix} element_1 \\ element_2 \\ element_3 \end{bmatrix} = \begin{bmatrix} node_1, node_2 \\ node_2, node_3 \\ node_3, node_4 \end{bmatrix} = \begin{bmatrix} id_1 \\ id_2 \\ id_3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 & 8 & 9 \\ 7 & 8 & 9 & 10 & 11 & 12 \end{bmatrix}$$

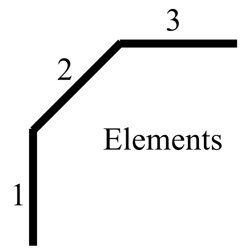
$$\mathbf{K}_a(id_i,\ id_i) \mathrel{+}= \mathbf{K}_g$$

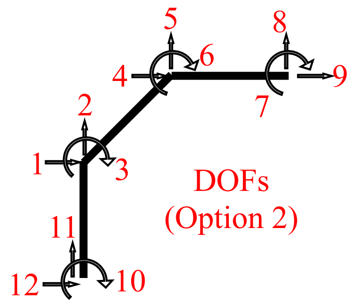
$$\mathbf{DOF} = \begin{bmatrix} node_1 \\ node_2 \\ node_3 \\ node_4 \end{bmatrix} = \begin{bmatrix} 12 & 11 & 10 \\ 1 & 2 & 3 \\ 4 & 5 & 6 \\ 9 & 8 & 7 \end{bmatrix}$$

$$\mathbf{ID} = \begin{bmatrix} 12 & 11 & 10 & 1 & 2 & 3 \\ 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 9 & 8 & 7 \end{bmatrix}$$

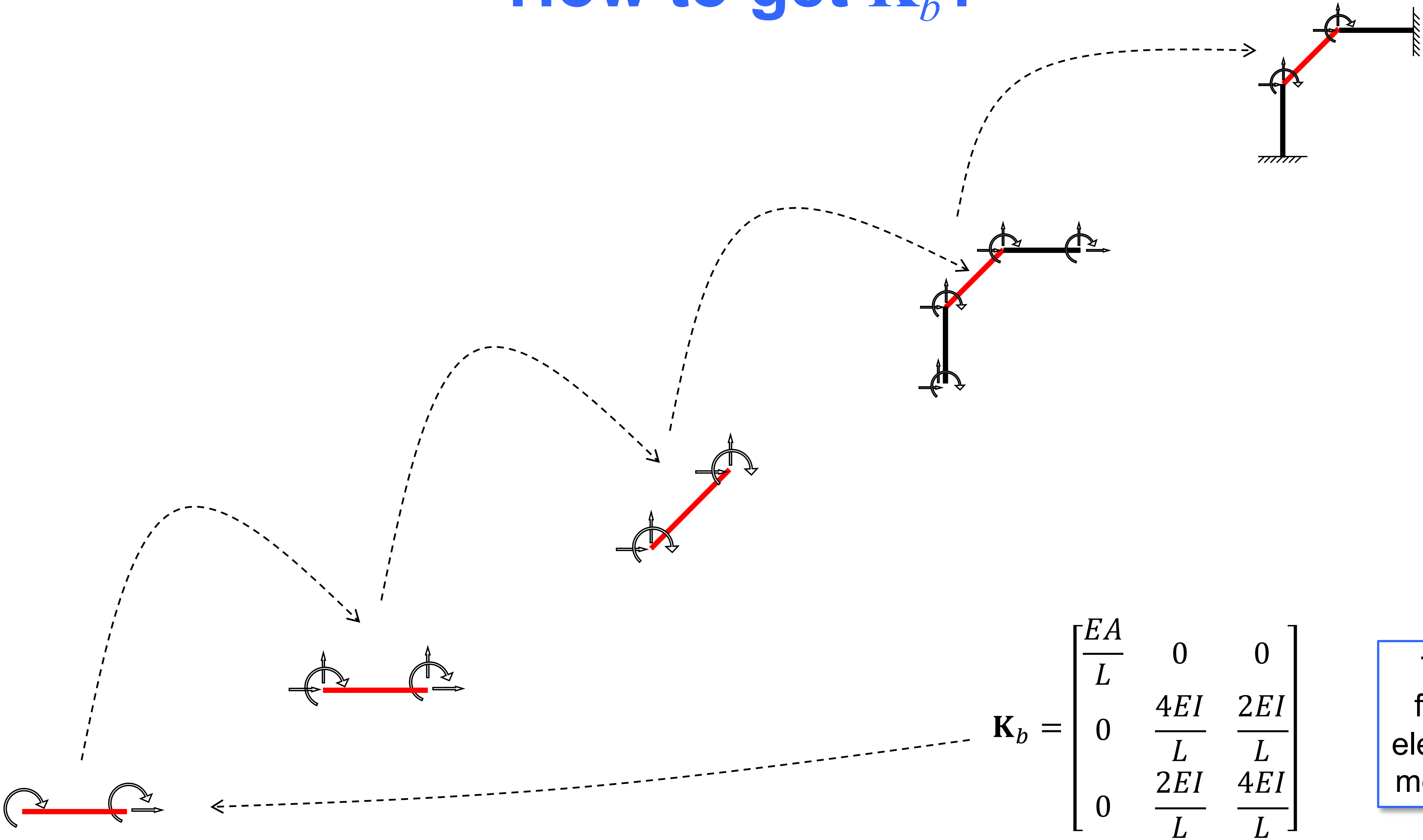


# How to get $\mathbf{K}_b$?



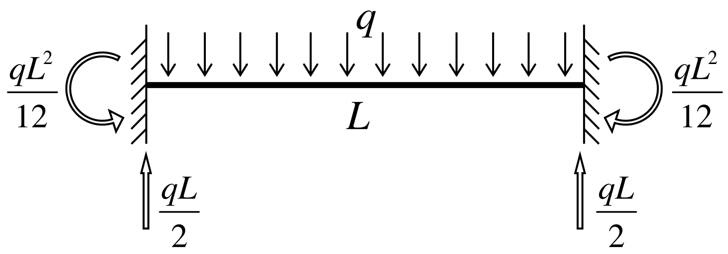
$$\mathbf{K}_b = \begin{bmatrix} \frac{EA}{L} & 0 & 0 \\ 0 & \frac{4EI}{L} & \frac{2EI}{L} \\ 0 & \frac{2EI}{L} & \frac{4EI}{L} \end{bmatrix}$$

The finite element method!

# Load Vector

**Ku=F**

$$\mathbf{K}\mathbf{u} = \grave{\mathbf{F}} - \bar{\mathbf{F}} = \mathbf{F}$$




$$\bar{\mathbf{F}}_b = \begin{Bmatrix} 0 \\ -\dfrac{qL^2}{12} \\ \dfrac{qL^2}{12} \end{Bmatrix}$$

$$\bar{\mathbf{F}}_f = \mathbf{T}_{af}^{\mathrm{T}} \left( \sum_{i=1}^{numEl} \mathbf{T}_{ga,i}^{\mathrm{T}} \mathbf{T}_{lg,i}^{\mathrm{T}} \mathbf{T}_{bl,i}^{\mathrm{T}} \bar{\mathbf{F}}_{b,i} \right)$$

# Solution & Element Forces

Never: $\mathbf{u}_f = \mathbf{K}_f^{-1}\,\mathbf{F}_f$

Always: $\mathbf{u}_f = \mathrm{Solve}(\mathbf{K}_f, \mathbf{F}_f)$

$$\mathbf{u}_b = \mathbf{T}_{bl}\mathbf{T}_{\mathrm{lg}}\mathbf{T}_{ga}\mathbf{T}_{af}\mathbf{u}_f$$

$$\grave{\mathbf{F}}_b = \mathbf{K}_b\mathbf{u}_b + \bar{\mathbf{F}}_b \quad \Longleftrightarrow \quad M_{NF} = \frac{2EI}{L}\cdot\left(2\cdot\theta_N + \theta_F - 3\cdot\psi_{AB}\right) + FEM_{NF}$$

# Additional Topics

Static condensation

Settlements & imposed displacements

Springs

DOF dependencies

More lectures:

Terje’s Toobox:

terje.civil.ubc.ca